

A CATEGORICAL

ACCOUNT OF THE LOCALIC

CLOSED SUBGROUP THEOREM

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TO APPEAR: CONN. MATH. UNIV. CAROL.

MOTIVATION

$\mathbb{Q} \subseteq \mathbb{R}$, \mathbb{Q} NOT CLOSED

$Loc \equiv (Frames)^{op}$
frame = cHa, $\mathbb{R}X$

IN CONTRAST TO TOPOLOGY, IN LOCALE THEORY, EVERY LOCALIC SUBGROUP IS CLOSED. [ISBELL, KŘÍŽ, PULTR, ROSICKÝ]

SO, WHILST LOCALE THEORY LOOKS LIKE TOPOLOGY, IT IS FUNDAMENTALLY DIFFERENT

STONE-ČECH, COMPACT HAUSDORFF, VIETORIS

RELATIONSHIP BETWEEN ALGEBRAIC STRUCTURE & PURELY TOPOLOGICAL

QUESTION

- CATEGORICAL APPROACHES EXIST TO LOCALE THEORY, RESULT STILL HOLD?

\mathcal{C} ORDER ENRICHED
 $\exists \$ \in \text{Ob}(\mathcal{C}), \mathbb{P}X \leftarrow \X

- THESE APPROACHES ARE ORDER DUAL; WHAT IS DUAL?

ALSO FOUND

- HOFMANN - MIBLOVE THEOREM KEY STEP
- NON-STANDARD DEFINITION OF DENSE.

LOCALE THEORY

LOC ORDER ENRICHED, FINITE LIMITS

$\$$ SIERPIŃSKI LOCALE, IS ORDER INTERNAL DLAT.

$$\Delta \dashv \Pi, \sqcup \dashv \Delta$$

$\$$ CLASSIFIES CLOSED & OPEN SUBOBJECTS.

VICKERS/TOWNSEND FOR ANY FRAMES Ω_X & Ω_Y ,

$$\text{SUP}(\Omega_X, \Omega_Y) \cong \sqcup\text{-SLAT}[\$, \$^X, \$^Y] \text{ IN } [\text{Loc}^{\text{OP}}, \text{Set}]$$

\vee PRESERVING

$$\text{PREFR}(\Omega_X, \Omega_Y) \cong \Pi\text{-SLAT}[\$, \$^X, \$^Y]$$

$$\$^X \text{ IS } (\text{Y}\$)^{\text{YX}}$$

\vee & \wedge PRESERVING

IDEA. ANY DEFINITION/FACT IN LOCALE THEORY THAT INVOLVES INTERACTION BETWEEN SUP-LATTICE HOMOMORPHISMS AND LOCALE MAPS CAN BE REPLICATED IN ANY \mathcal{L} WITH $\$ \in \text{DLAT}(\mathcal{L})$.

ASIDE

$$[\mathcal{L}^{\text{OP}}, \text{Set}]$$



$$\$^X \cong \$^Y \Rightarrow X \cong Y$$

$$x_0 \mapsto x \Rightarrow \$^x \xrightarrow{\$^i} \x_0$

FACTS ABOUT LOC.

$\exists \mathcal{P}_L$, LOWER POWER MONAD.
IT IS KZ.
 $Loc(Y, \mathcal{P}_L X) \cong SUP(\Omega X, \Omega Y)$

$\Omega(X \times Y) \cong \Omega X \otimes_{SUP} \Omega Y$

AXIOMS ON \mathcal{C}

$\exists \mathcal{P}_L$, A KZ MONAD ON SUCH THAT,
 $Loc(Y, \mathcal{P}_L X) \cong U-SAT[\$^X, \$^Y]$

$$\begin{matrix} \$^X \times \$^Y & \xrightarrow{\pi_1, \pi_2} & \$^X & \times & \$^Y & \xrightarrow{\pi} & \$^{X \times Y} \end{matrix}$$
 IS UNIVERSAL ~~U~~ U-BILINEAR

DEFINITIONS

$f: X \rightarrow Y$ OPEN IF THERE EXISTS
 $\exists f: \Omega X \rightarrow \Omega Y$, $\exists f \vdash \Omega f$ AND
 $\exists f(a \wedge b) = \exists f a \wedge b$
FOR ALL $a \in \Omega X$, $b \in \Omega Y$

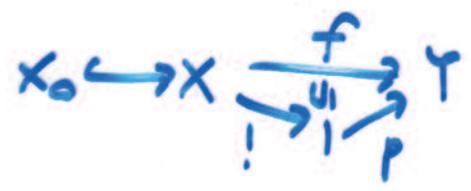
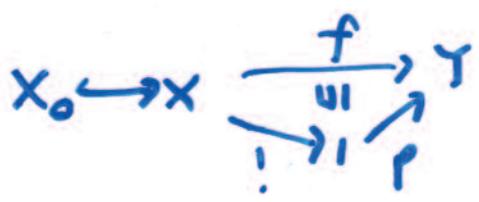
OPEN
 $f: X \rightarrow Y$ IF $\exists f: \$^X \rightarrow \Y
 $\exists f \vdash \$f$ WITH
 $\exists f \times Id$
$$\begin{matrix} \$^X \times \$^Y & \xrightarrow{\exists f \times Id} & \$^{X \times Y} \\ \downarrow Id \times f & & \downarrow \pi \\ \$^X \times \$^X & \xrightarrow{\pi} & \$^X \xrightarrow{\exists f} \$^Y \end{matrix}$$

X OPEN IF $! : X \rightarrow 1$
OPEN (\exists_x NOT $\exists!$)

X OPEN IF $! : X \rightarrow 1$
OPEN.

$X_0 \hookrightarrow X$ FITTED IF

$X_0 \hookrightarrow X$ FITTED IF



THEOREMS

- OPEN SUBOBJECTS FITTED
- OPEN MAP PULLBACK STABLE

FACTS ABOUT LOC

$\exists P_U$, UPPER POWER MONAD.

IT IS COKZ.

$$\text{Loc}(Y, P_U X) \cong \text{PREF}(\Omega X, \Omega Y)$$

$$\Omega(X \times Y) \cong \Omega X \otimes_{\text{PREF}} \Omega Y$$

AXIOMS ON \mathcal{C}

(4)^{co}

P_U , ALIKE MONAD SUCH THAT

$$\mathcal{C}(Y, P_U X) \cong \Pi\text{-SLAT}[\$, \$]$$

$$\$^X \times \$^Y \xrightarrow{\pi_1, \pi_2} \$^{X \times Y} \xrightarrow{\cup} \$^{X \times Y}$$

IS UNIVERSAL ~~W~~ Π -BILINEAR.

DEFINITIONS

$f: X \rightarrow Y$ PROPER IF THERE EXISTS

$\forall_f: \Omega X \rightarrow \Omega Y$ PREFRAME HOM.

$$\Omega f \uparrow \forall_f \& \forall_f(a \vee \Omega f b) = \forall_f a \vee b$$

FOR ALL $a \in \Omega X, b \in \Omega Y$.

$f: X \rightarrow Y$ PROPER IF \exists

$\forall_f: \$^X \rightarrow \$^Y, \Omega f \uparrow \forall_f \&$

$$\$^X \times \$^Y \xrightarrow{\forall_f \times \text{id}} \$^Y \times \Y$

$$\begin{array}{ccc} \downarrow \text{id} \times \Omega f & & \downarrow \cup \\ \$^X \times \$^X & \xrightarrow{\cup} & \$^X \xrightarrow{\forall_f} \$^Y \end{array}$$

~~[scribble]~~

X COMPACT IF $! : X \rightarrow 1$ PROPER.

X COMPACT IF $! : X \rightarrow 1$ PROPER

$X_0 \hookrightarrow X$ WEAKLY CLOSED IF

$$X_0 \hookrightarrow X \xrightarrow{f} Y \begin{array}{c} \uparrow \pi_1 \\ \downarrow \pi_2 \\ \downarrow ! \end{array}$$

$X_0 \hookrightarrow X$ WEAKLY CLOSED

IF $X_0 \hookrightarrow X \xrightarrow{f} Y \begin{array}{c} \uparrow \pi_1 \\ \downarrow \pi_2 \\ \downarrow ! \end{array}$

THEOREMS

- CLOSED SUBOBJECTS WEAKLY CLOSED
- PROPER MAPS ARE PULLBACK STABLE.

THEOREM. AXIOMATICALLY ON \mathcal{C}

(6)

$$\mathcal{C}(1, P_0 X) \cong \{ X_0 \xrightarrow{i} X \mid X_0 \text{ COMPACT, } i \text{ FITTED} \}$$

$$\mathcal{C}(1, P_L X) \cong \{ X_0 \xrightarrow{i} X \mid X_0 \text{ OPEN, } i \text{ WEAKLY CLOSED} \}$$

(HOFMANN-MISLOVE).

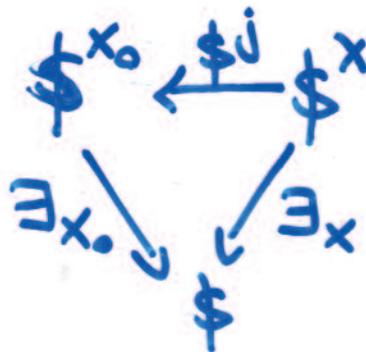
APPLICATION: ANY $X_0 \xrightarrow{i} X$ IN \mathcal{C} WITH X_0 OPEN FACTORS UNIQUELY AS

$$X_0 \xrightarrow{j} \bar{X}_0 \xrightarrow{\bar{i}} X$$

j DENSE

\bar{i} WEAKLY CLOSED

X_0, \bar{X}_0 OPEN $\&$



PROOF

$$X_0 \text{ OPEN} \Rightarrow \$^X \xrightarrow{\$^i} \$^{X_0} \xrightarrow{\exists x_0} \$ \quad \text{U-SLAT HOM.}$$

\Rightarrow THERE EXISTS $P: 1 \rightarrow P_L X$

CONSTRUCT

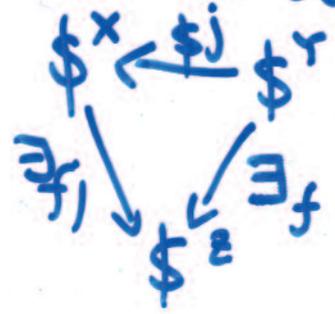
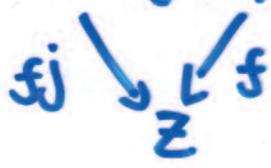
$$\bar{X}_0 \xrightarrow{\bar{i}} X \xrightarrow{\exists x} P_L X$$

$\downarrow \exists ! \rightarrow P$

$\text{Loc}_{\text{Sh}(Z)} \approx \text{Loc}/Z$, SIMILARLY \mathbb{C} .

RELATIVE DENSENESS.

GIVEN $X \xrightarrow{f} Y$ IN \mathbb{C} , j IS DENSE OVER f IF f OPEN & fj OPEN &

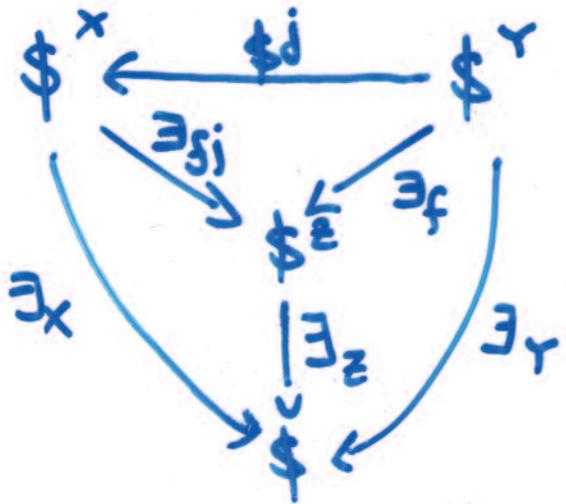


COMMUTES.

$Z=1$

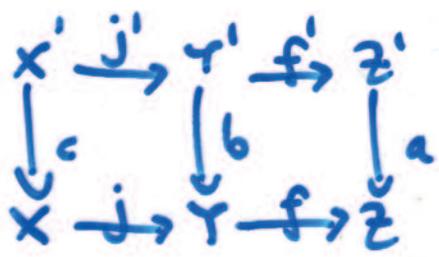
LEMMA IF j IS DENSE OVER f AND Z IS OPEN
 $\Rightarrow j$ IS DENSE.

PROOF



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LEMMA GIVEN PULLBACKS
 IF j DENSE OVER f THEN
 j' DENSE OVER f' .



FURTHER: THE INTERSECTION OF TWO RELATIVELY DENSE SUBOBJECTS IS RELATIVELY DENSE.

PROOF: BY PULLBACK STABILITY OF OPEN MAPS & BECK-CHEVALLEY. //

CLOSED SUBGROUP THEOREM FOR \mathbb{C} .

THEOREM: $H \xrightarrow{i} G$ A SUBGROUP AND H OPEN
THEN i WEAKLY CLOSED.

PROOF FACTOR i AS $H \xrightarrow{j} \bar{H} \xrightarrow{\bar{i}} G \dots$ HM \Rightarrow \bar{H} OPEN.
THEN SHOW j ISOMORPHISM.

KEY IS TO CONSIDER PULLBACK SQUARES

$$\begin{array}{ccccc}
 H \times H & \xrightarrow{j \times \text{id}} & \bar{H} \times \bar{H} & \xrightarrow{\bar{i} \times \bar{i}} & \bar{H} \\
 \pi_1 \downarrow & & \downarrow \pi_1 & & \downarrow ! \\
 H & \xrightarrow{j} & \bar{H} & \xrightarrow{!} & I
 \end{array}$$

THEN $j \times \text{id}$ DENSE OVER $\pi_2 \Rightarrow j \times \text{id}$ DENSE
SYMMETRICALLY $\text{id} \times j$ DENSE $\Rightarrow j \times j$ DENSE

SO CONSIDER

$$\begin{array}{ccccc}
 H \times H & \xrightarrow{j \times j} & \bar{H} \times \bar{H} & \xrightarrow{\bar{i} \times \bar{i}} & G \times G \\
 *_{H} \downarrow & & \downarrow *_{\bar{H}} & & \downarrow *_{G} \\
 H & \xrightarrow{j} & \bar{H} & \xrightarrow{\bar{i}} & G
 \end{array}$$

ALSO

$$\begin{array}{ccccc}
 H & \xrightarrow{j} & \bar{H} & \xrightarrow{\bar{i}} & G \\
 \text{inv}_H \downarrow & & \downarrow & & \downarrow \text{inv}_G \\
 \bar{H} & \xrightarrow{j} & \bar{H} & \xrightarrow{\bar{i}} & G
 \end{array}$$

i.e. \bar{H} IS A SUBGROUP OF G .

NEXT CONSIDER SAME PULLBACK SQUARE: ⑨

$$\begin{array}{ccccc}
 H \times H & \xrightarrow{j \times \text{Id}} & \bar{H} \times \bar{H} & \xrightarrow{\pi_2} & \bar{H} \\
 \pi_1 \downarrow & & \downarrow \pi_1 & & \downarrow ! \\
 H & \xrightarrow{j} & \bar{H} & \xrightarrow{!} & \mathbb{1}
 \end{array}$$

$\Rightarrow j \times \text{Id}$ DENSE ON π_2

BUT

$$\begin{array}{ccc}
 \bar{H} \times \bar{H} & \xrightarrow{\cong} & \bar{H} \times \bar{H} \\
 *_{\bar{H}} \searrow & & \swarrow \pi_2 \\
 & \bar{H} &
 \end{array}$$

$(h_1, h_2) \mapsto (h_1, h_1, h_2)$
 $(h_1, h_1, h_2) \mapsto (h_1, h_2)$

AS \bar{H} SUBGROUP $\Rightarrow j \times \text{Id}$ DENSE OVER $*_{\bar{H}}$
 & SYMMETRICALLY $\text{Id} \times j$ DENSE OVER $*_{\bar{H}}$
 $\Rightarrow j \times j$ DENSE OVER $*_{\bar{H}}$.

NEXT CONSIDER

$$\begin{array}{ccc}
 \begin{array}{ccc}
 H \times H & \xrightarrow{j \times j} & \bar{H} \times \bar{H} \\
 *_{\bar{H}} \downarrow & & \downarrow *_{\bar{H}} \\
 H & \xrightarrow{j} & \bar{H}
 \end{array} & \Rightarrow & \begin{array}{ccc}
 \$_{H \times H} & \xleftarrow{\$_{j \times j}} & \$_{\bar{H} \times \bar{H}} \\
 \$_{*_{\bar{H}}} \uparrow & & \downarrow \exists_{x_{\bar{H}}} \\
 \$_H & \xleftarrow{\$_j} & \$_{\bar{H}} \\
 & & \uparrow \exists_{*_{\bar{H}}(j \times j)}
 \end{array}
 \end{array}$$

$\Rightarrow \$^H \cong \$^{\bar{H}} \Rightarrow H \cong \bar{H}$.

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ORDER DUAL.

BOTH Loc & Loc^{co} SATISFY THE AXIOMS.

WHAT DOES THE RESULT LOOK LIKE FOR Loc^{co}?

" IF $H \leftrightarrow G$ IS A SUBGROUP WITH

H OPEN: THEN i IS WEAKLY
CLOSED IN Loc^{co} "

H COMPACT IN Loc IS FITTED IN Loc

NEW RESULT FOR LOCALE THEORY:

" EVERY COMPACT SUBGROUP IS FITTED. "

SUMMARY

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- * IN LOCALE THEORY ALL SUBGROUPS WITH OPEN DOMAIN ARE WEAKLY CLOSED
- * ORDER DUAL CATEGORICAL APPROACHES TO LOCALE THEORY EXIST.

* IN THESE APPROACHES ANY $X_0 \hookrightarrow X$ WITH X_0 OPEN FACTORS AS

$$X_0 \xrightarrow{j} \bar{X}_0 \xleftarrow{\bar{i}} X$$

WITH j DENSE, \bar{i} WEAKLY CLOSED.

- HOFMANN-MISLOVE

- NON-STANDARD DEF^N OF DENSE.

* THIS ALLOWS FOR A CATEGORICAL ^{PROOF} OF THE CLOSED SUBGROUP THEOREM.

* THERE IS AN ORDER DUAL:

- ALL COMPACT LOCALIC SUBGROUPS ARE FITTED.