AN AXIOMATIC ACCOUNT OF WEAK TRIQUOTIENT ASSIGNMENTS IN LOCALE THEORY

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Abstract. In locale theory, weak triquotient assignments on a map $f : X \longrightarrow Y$ can be represented as the points of the double power locale of f relative to the topos of sheaves over Y. A categorical proof of this representation theorem is given based on a categorical account of the Sierpiński locale.

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1. Introduction

We start by making some background comments about the category of locales (**Loc**) so as to be able to put the aims of the paper into an appropriate mathematical context. The category of locales is, by definition, the opposite of the category of frames; that is, the category whose objects are complete Heyting algebras and whose morphisms preserve arbitrary joins and finite meets. A locale morphism therefore points in the geometric direction which is opposite to the inverse image function f^{-1} determined by a continuous map in topology. One reason for studying **Loc** is that within it familiar topological notions persist and some results can be proved under milder than usual logical assumptions; the most well known example of this is the localic Tychonoff theorem which can be proved without using the axiom of choice, see [J82] for a textbook account. Locale theory has an established theory of hyperspaces, though this theory goes under the name 'powerlocales' (rather than 'hyperlocales') given its origins in the power domain constructions of Theoretical Computer Science. In contrast to the theory of hyperspaces, powerlocale constructions can be described categorically. For example, if X is a locale then the double power locale $\mathbb{P}(X)$ is given by the double exponential $\mathbb{S}^{\mathbb{S}^X}$ where $\mathbb{S}$ is the Sierpiński locale, see [VT04] for detail. The proof of this is constructive (in the sense of topos validity) and so can be carried out relative to the topos of sheaves, $Sh(Y)$, for any locale Y. Again in contrast to the topological situation, we have that the category of locales is slice stable, meaning that $\mathbf{Loc}_{Sh(Y)} \simeq \mathbf{Loc}/Y$, i.e. the category of locales relative to $Sh(Y)$ is equivalent to the slice of **Loc** at Y, [JT84]. In fact it follows from this equivalence and the constructive nature of our categorical description of the double power locale, that for any object Z_p of $\mathbf{Loc}/Y$ (i.e. for any locale map $p : Z \longrightarrow Y$),

$$\mathbb{P}_Y(Z_p) \cong \mathbb{S}_Y^{\mathbb{S}_Y^{Z_p}}$$

where $\mathbb{P}_Y$ denotes the double power locale construction and $\mathbb{S}_Y$ denotes the Sierpiński locale, both relative to $\mathbf{Loc}/Y$.

A less widely studied localic analog of a topological notion is that of triquotient assignment, see [P97] for the original insight and [T03] for the weakening that we will study here. Given a locale map $p : Z \longrightarrow Y$, a weak triquotient assignment on p is a morphism $p_\# : \mathbb{S}^Z \longrightarrow \mathbb{S}^Y$ that is required to interact with p in a certain manner (see Lemma 5.1). As established by Plewe, the study of triquotient assignments is relevant to the question of which epimorphisms are of effective descent in **Loc**.

In fact, there is a relationship between the double power locale construction and weak triquotient assignments so providing a connection between these two ideas: [T03] shows that weak triquotient assignments on a locale map $p : Z \longrightarrow Y$ are order isomorphic to the points of the double power locale $\mathbb{P}_Y(Z_p)$. This order isomorphism can be applied to prove standard pullback stability results for proper and open locale maps (see [Ver93] and [J80] for the results and [T03] for the application) so allowing these standard locale theoretic results to be seen as aspects of the study of weak triquotient assignments. The purpose of this paper is to give an axiomatic account of the relationship between weak triquotient assignments and the points of the double power locale. Our main result is to establish the order isomorphism

$$\mathcal{C}/Y(1, \mathbb{P}_Y(Z_p)) \cong \{p_\# : \mathbb{S}^Z \longrightarrow \mathbb{S}^Y : p_\# \text{ a w.t.a. on } p\}$$

for any morphism $p : Z \longrightarrow Y$ of a category $\mathcal{C}$ suitably axiomatised so as to behave like the category of locales.

Key to the axiomatisation is our definition of $\mathbb{S}$, a *Sierpiński object*, from which the double power locale at X can be defined as the double exponential: $\mathbb{P}(X) = \mathbb{S}^{\mathbb{S}^X}$. The axiomatisation is slice stable: if $\mathbb{S}$ is a Sierpiński object relative to $\mathcal{C}$ then $\mathbb{S}_Y$ (i.e. the morphism $\pi_1 : Y \times \mathbb{S} \longrightarrow Y$) is a Sierpiński object relative to $\mathcal{C}/Y$. Slice stability is key to the proof of our main result.

We assume familiarity with locale theory, e.g. [J82] and [JT84], including the representation of dcpo homomorphisms between frames as natural transformations in $[\mathbf{Loc}^{op}, \mathbf{Set}]$, see [VT04]. Note that locale theory is *order enriched* meaning that all universal constructions establish order isomorphisms and not just bijections between the relevant homsets. The internal lattices discussed are all *order* internal meaning that their finitary join(meet) operations are left(right) adjoint to the diagonal; for example the Sierpiński locale is an order internal distributive lattice in **Loc**.

Summary of contents

In the next section we list the axioms that are to be placed on a category $\mathcal{C}$. The theory in then built up in the following manner: firstly we check a categorical change of base result which is the known localic change of base adjunction when $\mathcal{C} = \mathbf{Loc}$. With this change of base result we then check that the axiomatic framework is stable under slicing. Next we derive some basic results about the Sierpiński object, essentially borrowing familiar techniques used to prove results about the subobject classifier in topos theory. This allows us to see that for the case $!^Z : Z \longrightarrow 1$, all maps $\mathbb{S}^Z \longrightarrow \mathbb{S}$ are weak triquotient assignments. Finally we apply change of base to show the main result, using the fact that all morphism $\mathbb{S}_Y^{Z_p} \longrightarrow \mathbb{S}_Y$ are weak triquotient assignments relative to Y. The last section discusses applications, outlining how familiar results from locale theory can be recovered by using the axiomatic framework.

2. Categorical axioms

We now present the axioms that are to be placed on a category $\mathcal{C}$; the proofs of each for the case $\mathcal{C} = \mathbf{Loc}$ are either explicit in the literature or easily obtained from known lattice theory.

Axiom 1. *$\mathcal{C}$ is an order enriched category with order enriched finite limits and finitary coproducts.*

Axiom 2. *For any objects X, Y and W in $\mathcal{C}/Z$, $X \times (Y + W) \cong X \times Y + X \times W$. Further $X \times 0 \cong 0$.*

Axiom 3. *There is an order internal distributive lattice denoted $\mathbb{S}$ such that, for $i : 1 \hookrightarrow \mathbb{S}$ equal to either $0_{\mathbb{S}}$ or $1_{\mathbb{S}}$, given a pullback square*

$$\begin{array}{ccc} a^*(i) & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow i \\ X & \xrightarrow{\ a\ } & \mathbb{S} \end{array}$$

a is uniquely determined by $a^*(i) \hookrightarrow X$.

We refer to $\mathbb{S}$ as a *Sierpiński object* if it satisfies this axiom. A variation of the axiom also, in effect, appears in [Tay00] via Definition 2.2. Note that $\mathbb{S}$ is not canonical so our underlying data is a pair $(\mathcal{C}, \mathbb{S})$; however we follow a usual convention, and just say '$\mathcal{C}$ is a category satisfying the axioms' rather than being explicit about which $\mathbb{S}$ is chosen. I have not been able to construct a category with multiple non-trivial Sierpiński objects.

In the next axiom for any object X of $\mathcal{C}$ we use the notation $\mathbb{S}^X$ for the functor

$$\begin{array}{ccc} \mathcal{C}^{op} & \longrightarrow & \mathbf{Set} \\ Y & \longmapsto & \mathcal{C}(Y \times X, \mathbb{S}) \end{array}$$

It can be verified, using Yoneda's lemma, that $\mathbb{S}^X$ is the exponential

$$\mathcal{C}(\_, \mathbb{S})^{\mathcal{C}(\_, X)}$$

in the presheaf category $[\mathcal{C}^{op}, \mathbf{Set}]$, so the notation is reasonable. We use the notation $\overline{\mathcal{C}}^{op}$ for the full subcategory of $[\mathcal{C}^{op}, \mathbf{Set}]$ consisting of all objects of the form $\mathbb{S}^X$. This category inherits an order enrichment from $\mathcal{C}$ and is closed under binary products by application of Axiom 2 ($\mathbb{S}^X \times \mathbb{S}^Y \cong \mathbb{S}^{X+Y}$ and $1 \cong \mathbb{S}^0$).

The next axiom reflects the relationship introduced as the 'double coverage theorem' in [VT04].

Axiom 4. *For any equalizer diagram*

$$E \xrightarrow{\ e\ } X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

in $\mathcal{C}$ the diagram

$$\mathbb{S}^X \times \mathbb{S}^X \times \mathbb{S}^Y \underset{\sqcap(Id \times \sqcup)(Id \times Id \times \mathbb{S}^g)}{\overset{\sqcap(Id \times \sqcup)(Id \times Id \times \mathbb{S}^f)}{\rightrightarrows}} \mathbb{S}^X \xrightarrow{\ \mathbb{S}^e\ } \mathbb{S}^E$$

is a coequalizer in $\overline{\mathcal{C}}^{op}$ where $\sqcap(Id \times \sqcup)$ is the composite

$$\mathbb{S}^X \times \mathbb{S}^X \times \mathbb{S}^X \xrightarrow{Id \times \sqcup} \mathbb{S}^X \times \mathbb{S}^X \xrightarrow{\ \sqcap\ } \mathbb{S}^X.$$

Note that Axiom 4 does not break the symmetry given by the order enrichment. A short calculation using the distributivity assumption on $\mathbb{S}$ shows that the composite $\sqcup(Id \times \sqcap)$ could have been used in the place of $\sqcap(Id \times \sqcup)$.

Axiom 5. *For any object X the double exponential $\mathcal{C}(\_, \mathbb{S})^{\mathbb{S}^X}$ exists in $[\mathcal{C}^{op}, \mathbf{Set}]$ and is representable.*

Following the notation for $\mathcal{C} = \mathbf{Loc}$ we use $\mathbb{P}(X)$ to denote the object that represents the double exponential $\mathcal{C}(\_, \mathbb{S})^{\mathbb{S}^X}$. The axiom is equivalent to requiring that $\mathbb{S}^{(\_)} : \mathcal{C} \longrightarrow \overline{\mathcal{C}}^{op}$ has a right adjoint.

It is clear that these axioms are stable under the order enrichment:

Theorem 2.1. *If an ordered enriched category $\mathcal{C}$ satisfies the axioms then so does its order dual, $\mathcal{C}^{co}$.*

Our next aim is to check that the axioms are slice stable and to do this we are going to need a change of base result relative to the axioms.

3. Axiomatic change of base

Let us recall how change of base works for the category of locales before we state and prove an axiomatic change of base result.

If $f : X \longrightarrow Z$ is a locale map then, by common abuse of notation,

$$f : Sh(X) \longrightarrow Sh(Z)$$

is a geometric morphism from the topos of sheaves over X to the topos of sheaves over Y. The direct image of f can be shown to preserve the property of being an internal dcpo and further defines a functor

$$f_* : \mathbf{dcpo}_{Sh(X)} \longrightarrow \mathbf{dcpo}_{Sh(Z)}.$$

[T03] shows that this functor has a left adjoint, $f^\#$, and so since every frame is a dcpo we have that for any frame $\Omega_{Sh(X)}(A)$ internal to $Sh(X)$ and any frame $\Omega_{Sh(Z)}(B)$ internal to $Sh(Z)$,

$$\mathbf{dcpo}_{Sh(X)}(f^\#\Omega_{Sh(Z)}(B), \Omega_{Sh(X)}(A)) \cong \mathbf{dcpo}_{Sh(Z)}(\Omega_{Sh(Z)}(B), f_*\Omega_{Sh(X)}(A)). \qquad (*)$$

Further it can be verified (i) that both f_* and $f^\#$ preserve the property of being a frame and of being a frame homomorphism and (ii) that the isomorphism $(*)$ preserves the property of being a frame homomorphism. Given these observations we have an adjunction

$$\mathbf{Fr}_{Sh(X)} \underset{f_*}{\overset{f^\#}{\rightleftarrows}} \mathbf{Fr}_{Sh(Z)}, \quad \bot$$

Now [JT84] shows that $\mathbf{Fr}_{Sh(X)} \simeq (\mathbf{Loc}/X)^{op}$ and from the details of the proof of this fact (e.g. C1.6 [J02]) it can be shown that

$$\Sigma_f : \mathbf{Loc}/X \longrightarrow \mathbf{Loc}/Z,$$

is equivalent to f_*^{op}, where Σ_f is the 'compose with f functor' i.e. $g \longmapsto f \circ g$. Hence, since pullback is by definition right adjoint to Σ_f, we have that $(f^\#)^{op}$ defines pullback. In other words change of base extends contravariantly to dcpo homomorphisms. This extra level of generality is key when discussing change of base axiomatically. Given that $f^\#$ is equivalent to pullback, taking $A \in \mathbf{Loc}/X$ and

$B \in \mathbf{Loc}/Z$ and re-interpreting dcpo homomorphisms between frames as natural transformations ([VT04]), (∗) reads

$$\overline{(\mathbf{Loc}/X)}^{op}(\mathbb{S}_X^{f^*B}, \mathbb{S}_X^A) \cong \overline{\mathbf{Loc}/Z}^{op}(\mathbb{S}_Z^B, \mathbb{S}_Z^{\Sigma_f A}).$$

In other words, for any locale map $f : X \longrightarrow Z$ the adjunction $\Sigma_f \dashv f^*$ embeds into an adjunction

$$\overline{\mathbf{Loc}/X}^{op} \underset{f_*}{\overset{f^\#}{\rightleftarrows}} \overline{\mathbf{Loc}/Z}^{op} \quad (\bot)$$

via $\mathbb{S}_Z^{(\_)} : \mathbf{Loc}/Z \longrightarrow \overline{\mathbf{Loc}/Z}^{op}$.

Our next lemma proves this axiomatically.

Lemma 3.1. *If $f : X \longrightarrow Z$ is a map in a category $\mathcal{C}$ that satisfies the axioms then the pullback adjunction $\Sigma_f \dashv f^*$ extends to an adjunction*

$$\overline{\mathcal{C}/X}^{op} \underset{f_*}{\overset{f^\#}{\rightleftarrows}} \overline{\mathcal{C}/Z}^{op} \quad (\bot)$$

via $\mathbb{S}_Z^{(\_)} : \mathcal{C}/Z \longrightarrow \overline{\mathcal{C}/Z}^{op}$.

Proof. Take $f^\#(\mathbb{S}_Z^B)$ to be the functor composition

$$(\mathcal{C}/X)^{op} \xrightarrow{\Sigma_f} (\mathcal{C}/Z)^{op} \xrightarrow{\mathbb{S}_Z^B} \mathbf{Set}$$

which, via a routine calculation, is naturally isomorphic to

$$(\mathcal{C}/X)^{op} \xrightarrow{\mathbb{S}_X^{f^*B}} \mathbf{Set}.$$

Take $f_*(\mathbb{S}_X^A)$ to be the functor composition

$$(\mathcal{C}/Z)^{op} \xrightarrow{f^*} (\mathcal{C}/X)^{op} \xrightarrow{\mathbb{S}_X^A} \mathbf{Set}$$

which, via a routine calculation, is naturally isomorphic to

$$(\mathcal{C}/Z)^{op} \xrightarrow{\mathbb{S}_Z^{\Sigma_f A}} \mathbf{Set}.$$

It is clear from the definition that the two squares

$$\begin{array}{ccc} \mathcal{C}/X & \longrightarrow & \overline{\mathcal{C}/X}^{op} \\ \Sigma_f \downarrow \ \uparrow f^* & & f_* \downarrow \ \uparrow f^\# \\ \mathcal{C}/Z & \longrightarrow & \overline{\mathcal{C}/Z}^{op} \end{array}$$

commute. So, for the claim of adjunction $f^\# \dashv f_*$, let $\eta : 1 \stackrel{\cdot}{\longrightarrow} f^*\Sigma_f$, $\epsilon : \Sigma_f f^* \stackrel{\cdot}{\longrightarrow} 1$ be the unit and counit of the adjunction $\Sigma_f \dashv f^*$. Then define $\overline{\eta} : Id \stackrel{\cdot}{\longrightarrow} f_* f^\#$, $\overline{\epsilon} : f^\# f_* \stackrel{\cdot}{\longrightarrow} Id$, by

$$\overline{\eta}_{\mathbb{S}_Z^B} = \mathbb{S}_Z^B \xrightarrow{\mathbb{S}_Z^{\epsilon_B}} \mathbb{S}_Z^{\Sigma_f f^* B}$$

for $B \in \mathcal{C}/Z$, and

$$\overline{\epsilon}_{\mathbb{S}_X^A} = \mathbb{S}_X^{f^*\Sigma_f A} \xrightarrow{\mathbb{S}_X^{\eta_A}} \mathbb{S}_X^A,$$

for $A \in \mathcal{C}/X$. The triangular identities for $\overline{\eta}$ and $\overline{\epsilon}$ are therefore immediate from the fact that they hold for ϵ and η. □

Note that by unwinding the definition of the extension of pullback adjunctions to natural transformations we have that for any morphism α of $\overline{\mathcal{C}/Y}^{op}$

$$[W_* l^{\#}(\alpha)]_X = \alpha_{\Sigma_l W^* X}$$

and

$$\alpha_{W_l} = [W_* l^{\#}(\alpha)]_1$$

for any morphism $l : W \longrightarrow Y$ and any object X of $\mathcal{C}$. Here, of course, we are using the notation $W^{\#} \dashv W_*$ for the extension of the pullback adjunction determined by $!^W : W \longrightarrow 1$ rather than, say, $(!^W)^{\#} \dashv (!^W)_*$.

The next lemma shows how the previous lemma specializes given the lattice structure assumed on $\mathbb{S}$.

Lemma 3.2. *Given $f : X \longrightarrow Z$, for $A \in \mathcal{C}/X$*

$$\begin{aligned} f_* \sqcap_{\mathbb{S}_X^A} &= \sqcap_{f_* \mathbb{S}_X^A},\ f_* 1_{\mathbb{S}_X^A} = 1_{f_* \mathbb{S}_X^A}, \\ f_* \sqcup_{\mathbb{S}_X^A} &= \sqcup_{f_* \mathbb{S}_X^A},\ f_* 0_{\mathbb{S}_X^A} = 0_{f_* \mathbb{S}_X^A}, \end{aligned}$$

and for $B \in \mathcal{C}/Z$

$$\begin{aligned} f^{\#} \sqcap_{\mathbb{S}_Z^B} &= \sqcap_{f^{\#} \mathbb{S}_Z^B},\ f^{\#} 1_{\mathbb{S}_Z^B} = 1_{f^{\#} \mathbb{S}_Z^B}, \\ f^{\#} \sqcup_{\mathbb{S}_Z^B} &= \sqcup_{f^{\#} \mathbb{S}_Z^B},\ f^{\#} 0_{\mathbb{S}_Z^B} = 0_{f^{\#} \mathbb{S}_Z^B}. \end{aligned}$$

Further if $f : X \longrightarrow Y$ is a morphism in $\mathcal{C}$ then $Y_*(\overline{\eta}_{\mathbb{S}_Y^{Z_p}}) = \mathbb{S}^{p^* f} : \mathbb{S}^Z \longrightarrow \mathbb{S}^{X \times_Y Z}$ for any $p : Z \longrightarrow Y$.

Proof. The assertions about f_* are immediate since f_* is a right adjoint and so preserves binary products. Recall that the meet and join operations are order internal and so are right and left adjoint to finite diagonals: if the finite diagonals are preserved so are their right/left adjoints provided it is the case, as we have here from construction, that the functor f_* preserves order.

For the assertions about $f^{\#}$ it is sufficient to verify that $f^{\#}$ preserves binary products. This follows from Axiom 2 since

$$\begin{aligned} f^{\#}(\mathbb{S}_Z^B \times \mathbb{S}_Z^B) &\cong f^{\#} \mathbb{S}_Z^{B+B} \cong \mathbb{S}_Z^{f^*(B+B)} \\ &\cong \mathbb{S}_Z^{f^* B + f^* B} \cong f^{\#}(\mathbb{S}_Z^B) \times f^{\#}(\mathbb{S}_Z^B). \end{aligned}$$

and similarly when $B = 0$.

For the further part, note that the unit of the adjunction $f^{\#} \dashv f_*$ is given by the counit of adjunction $\Sigma_f \dashv f^*$. This counit, at $p : Z \longrightarrow Y$, is the top arrow of

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{p^* f} & Z \\ {\scriptstyle f^* p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y. \end{array}$$

□

The point of this lemma is that the lattice structure associated with the Sierpiński object is stable under change of base. This observation together with our description of α_{W_l} in terms of W_* and $l^{\#}$ given above, are now applied to show that the axioms are slice stable.

Theorem 3.3. *If a category $\mathcal{C}$ satisfies the axioms then so does $\mathcal{C}/Y$ for any object Y of $\mathcal{C}$. Further if the pullback functor $Y^* : \mathcal{C} \longrightarrow \mathcal{C}/Y$ is required to preserve the Sierpiński object of $\mathcal{C}$ then there is a canonical Sierpiński object on $\mathcal{C}/Y$ such that the axioms are satisfied.*

Proof. That the first two axioms are slice stable is immediate. To see that the third axiom is slice stable note that it is easy to verify that $\mathbb{S}_Y$ (i.e. the map $\pi_1 : Y \times \mathbb{S} \longrightarrow Y$) satisfies the axiom relative to $\mathcal{C}/Y$. This choice for the Sierpiński object relative to Y becomes the canonical choice if we require Y^* to preserve the structure of $\mathcal{C}$.

Axiom 4. Say $E \overset{e}{\hookrightarrow} X_1 \underset{h_2}{\overset{h_1}{\rightrightarrows}} X_2$ is an equalizer diagram in $\mathcal{C}/Y$, we must show that for any $\alpha : \mathbb{S}_Y^{X_1} \longrightarrow \mathbb{S}_Y^Z$ for which

$$\begin{array}{rl} & \alpha \sqcap (Id \times \sqcup)(Id \times Id \times \mathbb{S}_Y^{h_1}) \\ = & \alpha \sqcap (Id \times \sqcup)(Id \times Id \times \mathbb{S}_Y^{h_2}) \quad \text{Eqn 1.} \end{array}$$

there exists unique $\beta : \mathbb{S}_Y^E \longrightarrow \mathbb{S}_Y^Z$ such that $\beta \mathbb{S}_Y^e = \alpha$. The first thing to note is that by application of the change of base proposition above we can assume that $Z = 1$.

So say we are given $\alpha : \mathbb{S}_Y^{X_1} \longrightarrow \mathbb{S}_Y$ satisfying Eqn 1. To define $\beta : \mathbb{S}_Y^E \longrightarrow \mathbb{S}_Y$ we must, for every $l : W \longrightarrow Y$, define a map

$$\mathcal{C}/Y(E \times_Y W_l, \mathbb{S}_Y) \longrightarrow \mathcal{C}/Y(W_l, \mathbb{S}_Y).$$

Since $\mathcal{C}/Y(E \times_Y W_l, \mathbb{S}_Y) \cong \mathcal{C}(E \times_Y W, \mathbb{S})$ and $\mathcal{C}/Y(W_l, \mathbb{S}_Y) \cong \mathcal{C}(W, \mathbb{S})$ this amounts to defining a map

$$\mathcal{C}(E \times_Y W, \mathbb{S}) \longrightarrow \mathcal{C}(W, \mathbb{S})$$

for each $l : W \longrightarrow Y$. Now if α satisfies Eqn 1 then $W_* l^{\#}(\alpha)$ satisfies

$$\begin{array}{rl} & (W_* l^{\#}(\alpha)) \sqcap (Id \times \sqcup)(Id \times Id \times \mathbb{S}^{h_1 \times Id}) \\ = & (W_* l^{\#}(\alpha)) \sqcap (Id \times \sqcup)(Id \times Id \times \mathbb{S}^{h_2 \times Id}). \end{array}$$

since, again by change of base, the extended functors W_* and $l^{\#}$ preserve the Sierpiński meet and join. Therefore, by Axiom 4, there exists a unique natural transformation $\gamma^{W_l} : \mathbb{S}^{E \times_Y W} \longrightarrow \mathbb{S}^W$ such that $\gamma^{W_l} \mathbb{S}^{e \times Id} = W_* l^{\#}(\alpha)$. Define $\beta : \mathbb{S}_Y^E \longrightarrow \mathbb{S}_Y$ by $\beta_{W_l} \equiv [\gamma^{W_l}]_1$. The construction of β from α is monotone so to complete the proof it remains to verify that β is natural, that $\beta \mathbb{S}_Y^e = \alpha$, and that if $\delta \mathbb{S}_Y^e = \alpha$ for some other natural transformation $\delta : \mathbb{S}_Y^E \longrightarrow \mathbb{S}_Y$ then $\delta = \beta$. These are all straight forward from construction, for completeness we prove that β is natural.

Say $n : W_l \longrightarrow V_m$ is a morphism in $\mathcal{C}/Y$ then by our description above of α_{W_l} the square

$$\begin{array}{ccc} \mathbb{S}^{V \times_Y X_1} & \xrightarrow{\Sigma_V m^*(\alpha)} & \mathbb{S}^V \\ {\scriptstyle \mathbb{S}^{n \times Id}} \downarrow & & \downarrow {\scriptstyle \mathbb{S}^n} \\ \mathbb{S}^{W \times_Y X_1} & \xrightarrow{\Sigma_W l^*(\alpha)} & \mathbb{S}^W \end{array}$$

commutes by naturality of α. However $\Sigma_V m^*(\alpha)$ factors as $\gamma^{V_m}\mathbb{S}^{e\times Id}$ and $\Sigma_W l^*(\alpha)$ factors as $\gamma^{W_l}\mathbb{S}^{e\times Id}$ and since $\mathbb{S}^{e\times Id}$ is an epimorphism we can conclude that

$$\begin{array}{ccc} \mathbb{S}^{V\times_Y E} & \xrightarrow{\gamma^{V_m}} & \mathbb{S}^V \\ \downarrow \mathbb{S}^{n\times Id} & & \downarrow \mathbb{S}^n \\ \mathbb{S}^{W\times_Y E} & \xrightarrow{\gamma^{W_l}} & \mathbb{S}^W \end{array}$$

commutes. By applying these natural transformations at 1 we therefore obtain the fact that β is natural.

Axiom 5. Firstly notice that for any objects X and Y of $\mathcal{C}$, the double exponential $\mathbb{S}_Y^{\mathbb{S}_Y^{X_Y}}$ exists in $[\mathcal{C}/Y^{op}, \mathbf{Loc}]$ and is representable. It is represented by $\mathbb{P}(X)_Y$ (i.e. $\pi_1 : Y \times \mathbb{P}(X) \longrightarrow Y$). This can be verified by change of base since, for any object W_l of $\mathcal{C}/Y$,

$$\begin{aligned} \mathcal{C}/Y(W_l, \mathbb{P}(X)_Y) &\cong \mathcal{C}(W, \mathbb{P}(X)) \\ &\cong Nat[\mathbb{S}^X, \mathbb{S}^W] \\ &\cong Nat[\mathbb{S}_Y^{X_Y}, \mathbb{S}_Y^{W_l}] \end{aligned}$$

where the last line is by change of base since $\mathbb{S}^W = W_*(\mathbb{S}_Y^{W_l})$.

Now any object X_f of $\mathcal{C}/Y$ occurs as an equalizer

$$X_f \xhookrightarrow{(f,Id)} X_Y \underset{\Delta\pi_1}{\overset{Id\times f}{\rightrightarrows}} Y_Y$$

and this gives rise, via Axiom 4 in the slice $\mathcal{C}/Y$, to a coequalizer in $\overline{\mathcal{C}/Y}^{op}$ which we can write as

$$\mathbb{S}_Y^{(X+X+Y)_Y} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathbb{S}_Y^{X_Y} \xrightarrow{\mathbb{S}_Y^{(f,Id)}} \mathbb{S}_Y^{X_f}.$$

If we therefore define $\mathbb{P}_Y(X_f)$ to be the equalizer of

$$\mathbb{P}(X)_Y \underset{\mathbb{S}_Y^{\beta}}{\overset{\mathbb{S}_Y^{\alpha}}{\rightrightarrows}} \mathbb{P}(X+X+Y)_Y$$

it clearly then has the right universal property of the double exponential.

□

4. The Sierpiński object

We make some observations that are consequences of the definition of $\mathbb{S}$ via Axiom 3. These types of results were initially observed by Taylor in [Tay00].

Lemma 4.1. *If $a, b : X \longrightarrow \mathbb{S}$ are two maps then*
(a) $\sqcap_{\mathbb{S}}(a,b)$ (open) classifies $a^*(1) \wedge_{Sub(X)} b^*(1)$ and
(b) $\sqcup_{\mathbb{S}}(a,b)$ (closed) classifies $a^*(0) \wedge_{Sub(X)} b^*(0)$.
The following are equivalent
(i) $a \sqsubseteq b$,
(ii) $a^*(1) \leq_{Sub(X)} b^*(1)$ and
(iii) $b^*(0) \leq_{Sub(X)} a^*(0)$.

From Axiom 3 any map with codomain $\mathbb{S}$ open classifies an *open subobject* and closed classifies a *closed subobject*, via pullback of $1, 0$ respectively. However, it seems overburdensome to distinguish the different types of classification and so the term "classifies" is used to cover either. Context will make it clear which is meant.

Proof. (a)

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ {\scriptstyle (1,1)}\downarrow & & \downarrow{\scriptstyle 1} \\ \mathbb{S}\times\mathbb{S} & \xrightarrow{\sqcap} & \mathbb{S} \end{array}$$

is a pullback since for any $p_1, p_2 \in \mathcal{C}(Z,\mathbb{S})$, $p_1 \sqcap_{\mathcal{C}(Z,\mathbb{S})} p_2 = 1_{\mathcal{C}(Z,\mathbb{S})} \Longrightarrow p_1 = 1_{\mathcal{C}(Z,\mathbb{S})}$ and $p_2 = 1_{\mathcal{C}(Z,\mathbb{S})}$. It is routine to verify that the pullback of $(1,1) : 1 \longrightarrow \mathbb{S}\times\mathbb{S}$ along $(a,b) : X \longrightarrow \mathbb{S}\times\mathbb{S}$ is $a^*(1) \wedge_{Sub(X)} b^*(1)$.

(b) Similarly since

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ {\scriptstyle (0,0)}\downarrow & & \downarrow{\scriptstyle 0} \\ \mathbb{S}\times\mathbb{S} & \xrightarrow{\sqcup} & \mathbb{S} \end{array}$$

is a pullback.

(i)$\Longleftrightarrow$(ii). $a^*(1) \leq_{Sub(X)} b^*(1)$ iff the subobject $a^*(1) \overset{i_{a^*(1)}}{\hookrightarrow} X$ factors via $b^*(1) \overset{i_{b^*(1)}}{\hookrightarrow} X$ which, by the definition of $b^*(1)$ as pullback, is the case iff $bi_{a^*(1)} = 1!^{a^*(1)}$. But $1!^{a^*(1)}$ is the top element of the external meet semilattice $\mathcal{C}(a^*(1),\mathbb{S})$ and so $a^*(1) \leq_{Sub(X)} b^*(1)$ iff $1!^{a^*(1)} \sqsubseteq bi_{a^*(1)}$.

Say $a \sqsubseteq b$, then since $1!^{a^*(1)} = ai_{a^*(1)}$, it follows that $1!^{a^*(1)} \sqsubseteq bi_{a^*(1)}$, and so $a^*(1) \leq_{Sub(X)} b^*(1)$.

Conversely say $a^*(1) \leq_{Sub(X)} b^*(1)$, then $bi_{a^*(1)} = 1!^{a^*(1)}$, and so the open subobject classified by $bi_{a^*(1)}$ (which is the pullback of $b^*(1)$ along $i_{a^*(1)}$, i.e. the meet of $a^*(1)$ and $b^*(1)$ in $Sub(X)$) is isomorphic to the subobject classified by $1!^{a^*(1)} = ai_{a^*(1)}$, i.e. $a^*(1) \xrightarrow{Id} a^*(1)$. In other words, $a^*(1) = a^*(1) \wedge b^*(1)$ in $Sub(X)$. But, $a^*(1) \wedge b^*(1)$ is classified by $\sqcap(a,b)$ and so $a = \sqcap(a,b)$, i.e. $a \sqsubseteq b$.

(i)$\Longleftrightarrow$(iii) follows by a symmetric argument. □

Lemma 4.2. *If $a, b : X \longrightarrow \mathbb{S}$ are two maps then the following are equivalent*

(i) $a \sqsubseteq b$,

(ii) for all $x_Z : Z \longrightarrow X$ if ax_Z factors through $1 : 1 \longrightarrow \mathbb{S}$ then bx_Z factors through $1 : 1 \longrightarrow \mathbb{S}$; and,

(iii) for all $x_Z : Z \longrightarrow X$ if bx_Z factors through $0 : 1 \longrightarrow \mathbb{S}$ then ax_Z factors through $0 : 1 \longrightarrow \mathbb{S}$.

Proof. Immediate from the fact that the lattice structure of open/closed subobjects agrees with the order enrichment in the manner shown in the previous lemma. □

As motivation for the next lemma it is worth first looking at some facts about the subobject classifier, Ω. For any frame ΩZ there is a unique frame homomorphism

$\Omega !^Z : \Omega \longrightarrow \Omega Z$; it is given by

$$i \longmapsto \bigvee^{\uparrow}_{\Omega Z} \{0_{\Omega Z}\} \cup \{1_{\Omega Z} \mid 1 \leq i\}.$$

From this it follows that for any function $a : \Omega Z \longrightarrow \Omega$, (and any $c \in \Omega Z$, $i \in \Omega$) the weakened Frobenius law,

$$a(c) \wedge i \leq a(c \wedge \Omega !^Z(i))$$

holds. If further a is a dcpo homomorphism then the weakened coFrobenius law,

$$a(c \vee \Omega !^Z(i)) \leq a(c) \vee i$$

holds. These facts follow since $i \leq j$ if and only if $i = 1$ implies $j = 1$, for truth values i, j in a topos (recall that $\Omega = P\{*\}$, i.e. the power set of the singleton set, and so these facts are just basic set theory). These properties are very particular to set theory, however they are also true in the abstract setting that we have with $\mathbb{S}$ in $\mathcal{C}$ taking the role of a 'subobject classifier' and natural transformations taking the role of dcpo homomorphisms:

Lemma 4.3. *If $\alpha : \mathbb{S}^Z \longrightarrow \mathbb{S}$ is a morphism in $\overline{\mathcal{C}}^{op}$ then*

(i) $\sqcap_{\mathbb{S}} (\alpha \times Id) \sqsubseteq \alpha \sqcap_{\mathbb{S}^Z} (Id \times \mathbb{S}^{!^Z})$

(ii) $\alpha \sqcup_{\mathbb{S}^Z} (Id \times \mathbb{S}^{!^Z}) \sqsubseteq \sqcup_{\mathbb{S}}(\alpha \times Id)$.

This lemma is essentially the same as Proposition 3.11 in [Tay00]. However here, in contrast, there is no assumption that $\mathbb{S}^Z$ exists as an object of $\mathcal{C}$. Topologically this means that there is no assumption that Z is locally compact.

Proof. (i) Given any object W and any $a : W \times Z \longrightarrow \mathbb{S}$, $b : W \longrightarrow \mathbb{S}$, it needs to be established that

$$[\sqcap_{\mathbb{S}}(\alpha \times Id)]_W(a, b) \sqsubseteq_{\mathcal{C}(W,\mathbb{S})} [\alpha \sqcap_{\mathbb{S}^Z} (1 \times \mathbb{S}^{!^Z})]_W(a, b).$$

The left hand side is

$$W \xrightarrow{(\alpha_W(a),b)} \mathbb{S} \times \mathbb{S} \xrightarrow{\sqcap_{\mathbb{S}}} \mathbb{S}$$

which (closed) classifies $[\alpha_W(a)]^*(1) \wedge_{Sub(W)} b^*(1)$. The right hand side is

$$\alpha_W(W \times Z \xrightarrow{(a,b\pi_1)} \mathbb{S} \times \mathbb{S} \xrightarrow{\sqcap_{\mathbb{S}}} \mathbb{S}).$$

So, by the previous lemma (part (i)), it is sufficient to prove that for any $x : X \longrightarrow W$ if $\sqcap_{\mathbb{S}}(\alpha_W(a), b)x = 1!^X$ then $\alpha_W(\sqcap_{\mathbb{S}}(a, b\pi_1))x = 1!^X$. Now if $\sqcap_{\mathbb{S}}(\alpha_W(a), b)x = 1!^X$, then $x : X \longrightarrow W$ factors through $(\alpha(a))^*(1)$ and $b^*(1)$, i.e. $[\alpha_W(a)]x = 1!^X$ and $bx = 1!^X$. By naturality of α,

$$\alpha_W(\sqcap_{\mathbb{S}}(a, b\pi_1))x = \alpha_X(\sqcap_{\mathbb{S}}(a, b\pi_1)(x \times Id)).$$

But,

$$\begin{aligned} & X \times Z \xrightarrow{x \times Id} W \times Z \xrightarrow{(a,b\pi_1)} \mathbb{S} \times \mathbb{S} \xrightarrow{\sqcap_{\mathbb{S}}} \mathbb{S} \\ = \; & X \times Z \xrightarrow{(a(x \times Id), bx\pi_1)} \mathbb{S} \times \mathbb{S} \xrightarrow{\sqcap_{\mathbb{S}}} \mathbb{S} \\ = \; & X \times Z \xrightarrow{(a(x \times Id), 1!^{X \times Z})} \mathbb{S} \times \mathbb{S} \xrightarrow{\sqcap_{\mathbb{S}}} \mathbb{S} \\ = \; & X \times Z \xrightarrow{a(x \times Id)} \mathbb{S} \end{aligned}$$

where the last line follows since $1!^{X\times Z}$ is the unit of the semilattice $\mathcal{C}(X \times Z, \mathbb{S})$. Therefore,

$$\begin{aligned}
\alpha_W(\sqcap_{\mathbb{S}}(a, b\pi_1))x &= \alpha_X(a(x \times Id)) \\
&= [\alpha_W(a)]x \qquad \text{(by naturality of } \alpha\text{)} \\
&= 1!^X.
\end{aligned}$$

(ii) The proof is entirely order dual, using (iii) of the previous lemma. □

5. Representation theorem for weak triquotient assignments

The following lemma exploits the distributivity assumption that we have placed on $\mathbb{S}$ and will provide a single equation characterisation for the definition of weak triquotient assignment to follow.

Lemma 5.1. *Given $p : Z \longrightarrow Y$ and a natural transformation $\alpha : \mathbb{S}^Z \longrightarrow \mathbb{S}^Y$ then*

$$\begin{array}{ccc}
\mathbb{S}^Z \times \mathbb{S}^Z \times \mathbb{S}^Y & \xrightarrow{Id \times Id \times \mathbb{S}^p} & \mathbb{S}^Z \times \mathbb{S}^Z \times \mathbb{S}^Z \\
\Big\downarrow {\scriptstyle (\pi_1, \pi_3, \sqcap(\pi_1, \pi_2))} & & \Big\downarrow {\scriptstyle \sqcap(Id \times \sqcup)} \\
\mathbb{S}^Z \times \mathbb{S}^Y \times \mathbb{S}^Z & & \mathbb{S}^Z \\
\Big\downarrow {\scriptstyle \alpha \times Id \times \alpha} & & \Big\downarrow {\scriptstyle \alpha} \\
\mathbb{S}^Y \times \mathbb{S}^Y \times \mathbb{S}^Y & \xrightarrow{\sqcup(\sqcap \times Id)} & \mathbb{S}^Y
\end{array}$$

commutes if and only if

$$\textit{(a)} \qquad \sqcap(\alpha \times Id) \sqsubseteq \alpha \sqcap (Id \times \mathbb{S}^p))$$

and

$$\textit{(b)} \qquad \alpha \sqcup (Id \times \mathbb{S}^p) \sqsubseteq \sqcup(\alpha \times Id).$$

Proof. Basic lattice theory exploiting the distributivity assumption. □

Definition 5.2. *If $p : Z \longrightarrow Y$ is a morphism in $\mathcal{C}$ then a* weak triquotient assignment *on p is a natural transformation $p_\# : \mathbb{S}^Z \longrightarrow \mathbb{S}^Y$ satisfying the conditions of the lemma.*

Lemma 4.3 is therefore showing that every map $\alpha : \mathbb{S}^Z \longrightarrow \mathbb{S}$ is a weak triquotient assignment on $!^Z : Z \longrightarrow 1$.

Weak triquotient assignments on locales were originally introduced as (very) weak triquotient assignments by Vickers in [V01] using different, but equivalent, equations. They are generalisations of Plewe's localic triquotient assignments, [P97], but are strictly weaker since the existence of one of our weak triquotient assignments on p does not imply that p is a surjection. In fact every map has two trivial triquotient assignments:

Example 5.3. *If $p : Z \longrightarrow Y$ is a map in $\mathcal{C}$ then*

$$\mathbb{S}^Z \longrightarrow 1 \xrightarrow{0} \mathbb{S}^Y$$

and

$$\mathbb{S}^Z \longrightarrow 1 \xrightarrow{1} \mathbb{S}^Y$$

are both weak triquotient assignments on p.

Work contained in [T03] shows that weak triquotient assignments on a locale map $p : Z \longrightarrow Y$ correspond to dcpo homomorphisms $\Omega_{Sh(Y)} Z_p \longrightarrow \Omega_{Sh(Y)}$. Lemma 4.3 proves this representation result axiomatically for $Y = 1$. We now provide an axiomatic account of this representation theorem for every Y.

Proposition 5.4. *Given an object Z_p in $\mathcal{C}/Y$ there is an order isomorphism between natural transformations $\mathbb{S}_Y^{Z_p} \longrightarrow \mathbb{S}_Y$ and weak triquotient assignments on p.*

Proof. For any $p : Z \longrightarrow Y$ in $\mathcal{C}/Y$ there is an equalizer diagram

$$Z_p \overset{(p,Id)}{\hookrightarrow} Z_Y \underset{\Delta\pi_1}{\overset{Id\times p}{\rightrightarrows}} Y_Y \qquad (*)$$

in $\mathcal{C}/Y$. By Axiom 4 relative to $\mathcal{C}/Y$ we need to show that for any map $\alpha : \mathbb{S}^Z \longrightarrow \mathbb{S}^Y \cong Y_*(\mathbb{S}_Y)$, α is a weak triquotient assignment for p if and only if its adjoint transpose under $Y^\# \dashv Y_*$ composes equally with the meet/join closure of the image of the fork $(*)$ under $\mathbb{S}_Y^{(\_)}$. In other words it must be verified that if $\alpha' : \mathbb{S}_Y^{Z_Y} \longrightarrow \mathbb{S}_Y$ is the adjoint transpose of α, then α is a weak triquotient assignment iff α' composes equally with

$$\mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Y_Y} \underset{Id \times Id \times (\mathbb{S}_Y^{\pi_1} \mathbb{S}_Y^{\Delta})}{\overset{Id \times Id \times \mathbb{S}_Y^{Id\times p}}{\rightrightarrows}} \mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Z_Y} \xrightarrow{\sqcap(Id \times \sqcup)} \mathbb{S}_Y^{Z_Y}. \qquad (**)$$

Now, α is a weak triquotient assignment iff the diagram

$$\begin{array}{ccc}
\mathbb{S}^Z \times \mathbb{S}^Z \times \mathbb{S}^Y & \xrightarrow{Id \times Id \times \mathbb{S}^p} & \mathbb{S}^Z \times \mathbb{S}^Z \times \mathbb{S}^Z \\
\downarrow{\scriptstyle (\pi_1, \pi_3, \sqcap(\pi_1, \pi_2))} & & \downarrow{\scriptstyle \sqcap(Id \times \sqcup)} \\
\mathbb{S}^Z \times \mathbb{S}^Y \times \mathbb{S}^Z & & \mathbb{S}^Z \\
\downarrow{\scriptstyle \alpha \times Id \times \alpha} & & \downarrow{\scriptstyle \alpha} \\
\mathbb{S}^Y \times \mathbb{S}^Y \times \mathbb{S}^Y & \xrightarrow{\sqcup(\sqcap \times Id)} & \mathbb{S}^Y
\end{array}$$

commutes. It has been observed already that $\sqcap_{\mathbb{S}_Y^{Z_Y}} = Y^\# \sqcap_{\mathbb{S}^Z}$ (Lemma 3.2) and clearly the pullback of $\mathbb{S}^p$ to Y is $\mathbb{S}_Y^{Id\times p}$. Therefore the adjoint transpose of the top and right hand part of this diagram is equal to the top row of $(**)$ postcomposed with α'. The proof will be completed provided it can be shown that the adjoint transpose of the left and bottom part of this diagram is equal to the bottom row of $(**)$ postcomposed with α'. Since $\sqcup_{\mathbb{S}^Y} = Y_*(\sqcup_{\mathbb{S}_Y})$ (see Lemma 3.2) the proof amounts to showing that the diagram

$$\begin{array}{ccc}
\mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Y_Y} & \xrightarrow{Id \times Id \times (\mathbb{S}_Y^{\pi_1} \mathbb{S}_Y^{\Delta})} & \mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Z_Y} \\
{\scriptstyle (\pi_1, \pi_3, \sqcap(\pi_1,\pi_2))} \downarrow & & \downarrow {\scriptstyle \sqcap(Id \times \sqcup)} \\
\mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Y_Y} \times \mathbb{S}_Y^{Z_Y} & & \mathbb{S}_Y^{Z_Y} \\
{\scriptstyle \alpha' \times \mathbb{S}_Y^{\Delta} \times \alpha'} \downarrow & & \downarrow {\scriptstyle \alpha'} \\
\mathbb{S}_Y \times \mathbb{S}_Y \times \mathbb{S}_Y & \xrightarrow{\sqcup(\sqcap \times Id)} & \mathbb{S}_Y
\end{array}$$

commutes. Noting that $\mathbb{S}_Y^{\pi_1} = \mathbb{S}_Y^{!^{Z_Y}}$, this follows by an application of Lemma 4.3 in the slice category $\mathcal{C}/Y$. □

The proof of our main result is now immediate:

Theorem 5.5. *For any morphism $p : Z \longrightarrow Y$ there is an order isomorphism*

$$\mathcal{C}/Y(1_Y, \mathbb{P}_Y(Z_p)) \cong \{p_\# : \mathbb{S}^Z \longrightarrow \mathbb{S}^Y : p_\# \text{ a w.t.a. on } p\}$$

Proof. This is clear from the proposition since by the construction of the double exponential $\mathbb{P}_Y(Z_p)$ relative to Y, morphisms $1_Y \longrightarrow \mathbb{P}_Y(Z_p)$ are order isomorphic to natural transformations $\mathbb{S}_Y^{Z_p} \longrightarrow \mathbb{S}_Y$. □

6. Applications

6.1. **Pullback stability of maps with weak triquotient assignments.** The following result was originally shown for locales by Vickers.

Proposition 6.1. *If $p : Z \longrightarrow Y$ is a morphism of $\mathcal{C}$ with some weak triquotient assignment $p_\#$ and there is a pullback diagram*

$$\begin{array}{ccc}
X \times_Y Z & \xrightarrow{\pi_2} & Z \\
{\scriptstyle \pi_1} \downarrow & & \downarrow {\scriptstyle p} \\
X & \xrightarrow{f} & Y
\end{array}$$

then there exists a unique weak triquotient assignment on π_1 such that Beck-Chevalley *holds; that is, such that $\mathbb{S}^f p_\# = (\pi_1)_\# \mathbb{S}^{\pi_2}$.*

Proof. The proof of this result given in [T03] for $\mathcal{C} = \mathbf{Loc}$ can be repeated more or less verbatim. $p_\#$s are in order isomorphism with natural transformations $\mathbb{S}_Y^{Z_p} \longrightarrow \mathbb{S}_Y$, but these last are clearly stable under change of base along f. □

6.2. **Pullback stability of open and proper locale maps.** Both proper and open maps can be characterised in terms of weak triquotient assignments. From the definitions of both of these classes of maps ([JT84] and [Ver93] respectively), a locale map $p : Z \longrightarrow Y$ is *open* if and only if it has a weak triquotient assignment $p_\#$ left adjoint to Ωp and it is *proper* if and only if it has a weak triquotient assignment $p_\#$ right adjoint to Ωp. Given our representation theorem we see that a locale map $p : Z \longrightarrow X$ is open (proper) if and only if there is a natural transformation $\mathbb{S}_X^{Z_p} \longrightarrow \mathbb{S}_X$ that is left(right) adjoint to $\mathbb{S}_X^{!_{Z_p}}$. Since the additional property of being left or right adjoint is clearly stable under change of base we have that open and proper maps are pullback stable from our more general result that maps with weak triquotient assignments are pullback stable. ([T03] also covers this specialization to open and proper maps in the case $\mathcal{C} = \mathbf{Loc}$.)

The important point about recalling these well known results is that they are now formally dual to each under the duality implied by the order enrichment (Theorem 2.1).

6.3. **Triquotient surjections are of effective descent.** Our techniques can be used to recover Plewe's result ([P97]) that localic triquotient surjections are of effective descent. A map $p : Z \longrightarrow Y$ of $\mathcal{C}$ is said to be a *triquotient surjection* if there exists a weak triquotient assignment $p_\#$ such that $p_\# \mathbb{S}^p = Id$; the usual notion is recovered when $\mathcal{C} = \mathbf{Loc}$. To prove effective descent of triquotient surjections the axiomatic framework needs to be strengthened slightly: $\mathcal{C}$ must have coequalizers of kernel pairs and the functor $\mathbb{S}^{(\_)}$ must reflect isomorphisms. Once these additional assumptions are in place it is easy to see that any triquotient surjection is the coequalizer of its kernel pair. The proof that, further, any triquotient surjection is of effective descent can proceed pretty much as in Plewe's original paper, [P97].

6.4. **Regularity of $\mathbf{KHaus}_\mathcal{C}$.** [T96] shows how it is possible to prove that the category of compact Hausdorff locales is regular using only formal properties of proper maps. We define $\mathbf{KHaus}_\mathcal{C}$ to be the full subcategory of $\mathcal{C}$ consisting of objects X such that both $! : X \longrightarrow 1$ and $\Delta : X \longrightarrow X \times X$ are proper; it follows that this category is regular. Since a map is proper relative to $\mathcal{C}$ if and only if it is open relative to $\mathcal{C}^{co}$, by Theorem 2.1 we have that compact Hausdorff objects can, relative to the axioms, be seen equally as objects with open finite diagonals, i.e. exactly the discrete objects. Therefore, up to the fragment of mathematics that can be developed using only regular logic, the theory of compact Hausdorff spaces and set theory are the same.

7. Summary

The axiomatic approach to locale theory explored in this paper is not canonical, so we are not claiming to have exactly isolated the correct categorical approach to locale theory. The approach does, however, lead to several foundational results and since some of these results cannot be expressed as formally dual without such an axiomatic framework, the work offers insight into how open and proper can be viewed as dual concepts. The approach has also been applied to other areas of locale theory, notably providing a categorical account of the Hofmann-Mislove theorem ([T05]) and the localic closed subgroup theorem ([T06]).

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