SCOTT IS NATURAL BETWEEN FRAMES

CHRISTOPHER TOWNSEND

ABSTRACT. Based on original work in [TV03], this paper gives a representation theorem for the Scott continuous maps between frames in terms of natural transformations between functors indexed by frames. The result specializes to frame homomorphisms, thereby giving a representation theorem for all locale maps (i.e. continuous maps as defined in locale theory).

1. INTRODUCTION

This work is a re-write of original material contained in [TV03]. The aim of this re-write is to express the results using the language of frame theory with less emphasis on locale theory, and using less of the detailed study of frame presentations as objects.

Recall that a frame is a complete lattice which satisfies the distributivity law

$$a \wedge \bigvee T = \bigvee \{a \wedge t \mid t \in T\}$$

for any element a and subset T. A frame homomorphism preserves arbitrary joins and finite meets, and so a category, **Fr**, of frames is defined. A frame is usually denoted ΩX, where X is the *corresponding locale*. This comes from the definition of the category of locales:

$$\mathbf{Loc} \equiv \mathbf{Fr}^{op},$$

that is, the category of locales is taken to be the dual of the category of frames. The category of locales is often considered as a good framework for constructive topology (see [J82]) and so a frame homomorphism $\Omega f : \Omega Y \to \Omega X$ (i.e. corresponding to a locale map $f : X \to Y$) is considered, in the localic context, exactly the data for a continuous map. The purpose of this paper is to give a representation theorem for these maps in terms of natural transformations.

A natural transformation is a map between functors (see [MacL71] for basic categorical definitions). Therefore to represent maps between frames as natural transformations we must first clarify what the functor is corresponding to any particular frame. Given a frame ΩX we define a functor $\mathbf{\Lambda}^{\Omega X} : \mathbf{Fr} \to \mathbf{Set}$ by

$$
\begin{aligned}
\mathbf{\Lambda}^{\Omega X}(\Omega Y) &= \Omega Y +_{\mathbf{Fr}} \Omega X \\
\mathbf{\Lambda}^{\Omega X}(\Omega g &: \Omega Y_1 \to \Omega Y_2) = \Omega g + 1_{\Omega X}
\end{aligned}
$$

where $+_{\mathbf{Fr}}$ denotes frame coproduct.

Directed complete partial orders (dcpos) are more general lattice structures than frames. Dcpos are defined as posets having joins for only directed subsets of elements. A dcpo homomorphism is required to preserve all directed joins; it is also known as a Scott continuous map. Our main theorem actually represents these more general homomorphisms:

Theorem 1.1. *For any two frames ΩX, ΩW there is a bijection between dcpo maps $\Omega X \to \Omega W$ and natural transformations $\mathbf{\Lambda}^{\Omega X} \overset{\cdot}{\to} \mathbf{\Lambda}^{\Omega W}$.*

The category of functors $\mathbf{Fr} \to \mathbf{Set}$ has finite products (they are calculated componentwise), and so the notion of an internal distributive lattice makes sense in this category. It can be shown that each $\mathbf{\Lambda}^{\Omega X}$ is an internal distributive lattice in $[\mathbf{Fr}, \mathbf{Set}]$ and since a frame homomorphism is exactly a dcpo homomorphism which is also a distributive lattice homomorphism, the following corollary, specializing the bijection of the theorem, seems reasonable.

Corollary 1.2. *There is a bijection between frame homomorphisms* $\Omega f : \Omega X \to \Omega W$ *and internal distributive lattice homomorphism* $\mathbf{\Lambda}^{\Omega X} \xrightarrow{\cdot} \mathbf{\Lambda}^{\Omega X}$.

The aim of this paper is to prove this theorem and corollary, thereby proving a representation theorem for dcpo homomorphisms between frames in terms of natural transformations and further showing that this representation specializes to frame homomorphisms (i.e. to locale maps).

The main results are then re-stated in terms of locales and finally a section has been included showing the implications of carrying out the main results relative to an arbitrary topos. It is shown that provided the topos is locally small, an external description of the set of internal dcpo homomorphisms is available.

1.1. Applications.

The broad application of this representation result is that it enables a new view to be taken of the notion of continuity. The infinitary aspects of continuity (directed joins) are taken care of by the 'higher order' structure contained in the definition of natural transformation, leaving only the finitary aspects to manipulate. The cost is that the finitary manipulations are now in the much larger category $[\mathbf{Fr}, \mathbf{Set}]$, but this category is relatively well behaved.

Specifically the result has application to the theory of power locales (which, very broadly speaking, is the localic study of hyperspaces). In [TV03] the result is used to show that the general points of the upper, lower and double power locales can all be described in terms of natural transformations. In particular it is shown that whereas the localic function space $\mathbb{S}^X$ will only exist as a locale if X is locally compact, with this representation theorem, we have that $\mathbb{S}^{\mathbb{S}^X}$ exists as a locale *for every* X. This is proved here as Corollary 5.6.

In [T04] the results are applied to the problem of axiomatizing the category of locales. By axiomatizing that natural transformations are related to continuous maps in the way that dcpo homomorphism are known to be related to locale maps, some properties of the category of locales can be proved axiomatically. For example, pullback stability of proper and open maps can be shown.

2. Dcpo Presentations

To get started with dcpo maps, a clear first step is to show that dcpo presentations present. By a *dcpo presentation* we shall mean a poset P (of *generators*) and a set of equations

$$\lambda(r) = \bigvee{}^{\uparrow}\{l \mid l\pi r\}$$

indexed by $r \in R$. So $\lambda : R \to P$ and $\pi \subseteq P \times R$ is such that $\{l \mid l\pi r\}$ is a directed subset of P for every $r \in R$. By 'presenting' we mean that $\mathbf{dcpo}\langle P \ qua \ poset \mid R\rangle$ is well defined, i.e. there exists a monotone map $k : P \to \mathbf{dcpo}\langle P \ qua \ poset \mid R\rangle$ satisfying the equations R such that for every monotone map $h : P \to A$, with A a dcpo and h satisfying R, there exists a unique dcpo homomorphism $f : \mathbf{dcpo}\langle P \ qua \ poset \mid R\rangle \to A$ such that $f \circ k = h$.

That dcpo presentations present seems to be folklore. It is equivalent to the fact that coequalizers of dcpos exist, and this has been known at least since [M77]. We now give a proof that re-applies the techniques of [JV91], where preframe presentations are proved to present from the fact that frame presentations do. Here, we replace preframes by dcpos and frames by suplattices.

The category of suplattices ($\mathbf{Sup}$) has as objects complete lattices and has as morphisms arbitrary join preserving maps. It was extensively studied in [JT84]. For example suplattice coequalizers exist: if M is a suplattice and R a subset of $M \times M$, then the set of R-coherent elements forms the quotient, where an $m \in M$ is R-coherent iff for every aRb it is the case that $a \leq m$ iff $b \leq m$. The quotienting map is $z : c \longmapsto \wedge\{m \mid m \ R\text{-coh.}, \ m \geq c\}$, which is left adjoint to the inclusion and so preserves arbitrary joins. To prove this quotienting claim, note that if $h : M \to Q$ is a suplattice homomorphism enjoying $h(a) = h(b)$ for every aRb then $h_* h(c)$ is R-coherent for every $c \in M$ where h_* is the right adjoint to h. It follows that the category of suplattices has coequalizers (take $R = \{(f(n), g(n) \mid n \in N\}$ for any $f, g : N \rightrightarrows M$ to be coequalized).

Lemma 2.1. *If A is a dcpo, then the free suplattice over it is provided by the set of Scott closed subsets.*

Proof. Let us write $F(A)$ for the set of Scott closed subsets of A (that is, the lower closed subsets which are closed under directed joins). Any intersection of Scott closed subsets is clearly Scott

closed and so $F(A)$ is a complete lattice. $\downarrow\colon A \to F(A)$ is Scott continuous (preserves directed joins). Now for any $B \in F(A)$, $B = \bigvee\{\downarrow b \mid b \in B\}$ since the join always contains the set theoretic union. So, given any dcpo morphism $\phi : A \to M$ with M a suplattice, the assignment $q(B) = \bigvee_M\{\phi(b) \mid b \in B\}$ is therefore necessary if ϕ is to factor as $q \downarrow$ with q a suplattice homomorphism. But $r : M \to F(A)$ given by $r(m) = \{b \mid \phi(b) \leq m\}$ provides a right adjoint to q so we know that q, so defined, is a suplattice homomorphism. Therefore $F(A)$ provides the correct universal properties. $\square$

Theorem 2.2. (dcpo presentations present) *For any dcpo presentation (P, R, λ, π), $\mathbf{dcpo}\langle P$ (qua poset)$\mid R\rangle$ is well defined.*

Proof. First note that the problem reduces to a proof of the existence of **dcpo** coequalizers since the ideal completion (i.e. the set of lower closed directed subsets) of any poset is the free dcpo on that poset (the universal map is $a \longmapsto \downarrow a$). We are reduced to finding the coequalizer of

$$idl(R) \rightrightarrows idl(P)$$

where $idl(\_)$ denotes the ideal completion and the arrows are determined by the equations of the presentation (and R is taken as a discrete poset, so $idl(R) \cong R$).

To find the dcpo coequalizer of $f, g : A \rightrightarrows B$, the first step is to take the suplattice coequalizer of Ff, Fg, giving a suplattice homomorphism $h' : F(B) \to C'$. Here F is the free construction as in the proof of the lemma. We therefore get a dcpo homomorphism $h' \circ \downarrow\colon B \to C'$. Next take the image factorization in **dcpo** to get $i \circ h : B \twoheadrightarrow C \hookrightarrow C'$. (The category of dcpos does have image factorizations: just take the intersection of all subdcpos containing the set theoretic image of the homomorphism to be factorized.)

h is the required dcpo coequalizer of f and g. If $k : B \to D$ composes equally with f and g, then $F(k)$ factors via C' as $k' \circ h'$ (say). Set $C_0 = \{c' \in C' \mid \exists d \in D, k'(c') =\downarrow d\}$. C_0 is a subdcpo of C' since k' and $\downarrow$ are dcpo homomorphisms. But $h' \circ \downarrow\colon B \to C'$ factors via C_0 ($k'h' \downarrow= F(k) \downarrow=\downarrow k$) and so $C \subseteq C_0$ by definition of image factorization. But k' restricts to a map from C_0 to D by construction, and so there is a dcpo homomorphism $C \to D$ as required, since for any $b \in B$, $k'h' \downarrow b =\downarrow k(b)$.

Uniqueness follows since h is a surjection (and so an epimorphism) in **dcpo** (as it is formed from image factorization). $\square$

Our next aim is to give a dcpo presentation for certain suplattice coequalizers. If we are given a dcpo presentation (L, R, λ, π) with L a join semilattice then R is *join stable* if for every $l' \in L$ and every equation

$$\lambda(r) = \bigvee^{\uparrow}\{l \mid l\pi r\}$$

in R it is also the case that

$$\lambda(r) \vee l' = \bigvee^{\uparrow}\{l \vee l' \mid l\pi r\}$$

is in R. With this definition we have,

Theorem 2.3. *let L be a join semilattice and R a join stable set of directed equations on it. Then*

$$\mathbf{Sup}\langle L \ (qua \ \vee \text{-}SemiLat) \mid R\rangle \cong \mathbf{dcpo}\langle L \ (qua \ poset) \mid R\rangle.$$

Proof. The right-hand side is defined; let us denote it by A. Now for any $l \in L$ the join stability assumption on R enables us to define a dcpo homomorphism $\phi_l : A \to A$ by

$$\phi_l(l') = l' \vee l.$$

Since $l' \le \phi_l(l')$ for any $l' \in L$, we have that $\phi_l(a)$ is an upper bound for a, for any $a \in A$ (since from the universal characterization of A it can be shown that every $a \in A$ is the directed join of $l \le a$, $l \in L$; consider the subdcpo of all elements of A for which this is true).

Now for each $a \in A$ define a dcpo homomorphism $\psi_a : A \to A$ by

$$\psi_a(l) = \phi_l(a).$$

To check this is well defined it must be verified that $\psi_a(\_) : L \to A$ is monotone and satisfies R. But $\psi_l(\_) : L \to A$ is monotone for every $l \in L$ and also satisfies R by the join stability assumption, and so the same is true for $\psi_a(\_) : L \to A$ since $\phi_l(\_)$ preserves directed joins and so $\psi_a(l)$ is a directed join of $\psi_{l'}(l)$ for any a.

Next, check that $\psi_c(c) = c$ for any $c \in A$ (since $c = \vee^{\uparrow}\{l \in L \mid l \le c\}$) and from this it follows that $\psi_b(a)$ is the binary join of a and b in A. The bottom element of L, 0_L has the property that $0_L \le l$ for every $l \in L$ and so $0_L \le a$ for every $a \in A$. Therefore A is a join semilattice (and also a dcpo) and so is a suplattice.

To prove that $A \cong \mathbf{Sup}\langle L \text{ (qua } \vee\text{-SemiLat)}| \ R\rangle$ we must show that every join semilattice homomorphism $\sigma : L \to M$ which satisfies R must factor uniquely as a suplattice homomorphism via A (where M is an arbitrary suplattice). By definition of A, $\sigma : L \to M$ factors uniquely as a dcpo homomorphisms, $\overline{\sigma} : A \to M$ say. To prove that $\overline{\sigma}$ is a suplattice homomorphism it must be verified, for example, that $\overline{\sigma}(\psi_b(a)) \leq \overline{\sigma}(a) \vee_M \overline{\sigma}(b)$ and this follows from the construction of ψ_b and the fact that $\sigma : L \to M$ is a join semilattice homomorphism. $\square$

2.1. **Dcpo presentations for Frame Coproduct.** It is known that frame coproduct (i.e. locale product) is the same thing as suplattice tensor (e.g. III, 2, of [JT84], but easily verified directly). Since we know that suplattice tensor can be defined (as it can be expressed as a coequalizer) it follows from the last theorem that we can now show how to describe frame coproduct via a dcpo presentation.

Proposition 2.4. *Suppose ΩX and ΩY are frames then*

$$\Omega X +_{\mathbf{Fr}} \Omega Y \cong \mathbf{dcpo}\langle \Omega X \otimes_{\vee \mathbf{Slat}} \Omega Y \ \text{(qua poset)} \mid R^{\otimes}_{\mathbf{dcpo}}\rangle$$

where $R^{\otimes}_{dcpo}$ consists of the equations

$$u \vee (\vee^{\uparrow} I \otimes b) = \bigvee\nolimits^{\uparrow} \{u \vee (i \otimes b) \mid i \in I\} \ and$$

$$u \vee (a \otimes \vee^{\uparrow} J) = \bigvee\nolimits^{\uparrow} \{u \vee (a \otimes j) \mid j \in J\}$$

taken over all (directed) $I \subseteq^{\uparrow} \Omega X$, $J \subseteq^{\uparrow} \Omega Y$ all $(a,b) \in \Omega X \times \Omega Y$ and all $u \in \Omega X \otimes_{\vee SLat} \Omega Y$.

Proof. It has been observed that $\Omega X +_{\mathbf{Fr}} \Omega Y \cong \Omega X \otimes_{\mathbf{Sup}} \Omega Y$. Therefore by splitting the suplattice bilinearity into finite join semilattice bilinearity and directed join (i.e. dcpo) bilinearity we have the first line of

$$\Omega X \otimes_{\mathbf{Sup}} \Omega Y \cong \mathbf{Sup}\langle \Omega X \otimes_{\vee \mathbf{Slat}} \Omega Y \ \text{(qua } \vee Slat)|$$

$$(\vee^{\uparrow} I \otimes b) = \bigvee\nolimits^{\uparrow}_{I} i \otimes b, \ (a \otimes \vee^{\uparrow} J) = \bigvee\nolimits^{\uparrow}_{J} a \otimes j\rangle$$

$$\cong \mathbf{Sup}\langle \Omega X \otimes_{\vee \mathbf{Slat}} \Omega Y \ \text{(qua } \vee Slat)|R^{\otimes}_{\mathbf{dcpo}}\rangle$$

where the second line follows since adding the $u \in \Omega X \otimes_{\vee \mathbf{SLat}} \Omega Y$ into the equations clearly has no effect on the suplattice being presented. But by construction $R^{\otimes}_{\mathbf{dcpo}}$ is join stable and so this result follows from the last theorem. $\square$

Given the join semilattice tensor used above it will be helpful to have the following explicit description of it for use later. Here $\mathcal{F}$ denotes the (Kuratowski) finite powerset construction and **Pos** is the category of posets with monotone maps.

Proposition 2.5. *Let C and D be two join semilattices. Then their join semilattice tensor $(C \otimes_{\vee \mathbf{Slat}} D)$ is given by*

$$\mathbf{Pos}\langle \mathcal{F}(C \times D) \text{ (qua poset)} \,|$$
$$\{(\vee_{k \in K} c_k, \vee_{l \in L} d_l)\} \cup I = \{(c_k, d_l) | k \in K, l \in L\} \cup I \rangle,$$

where the equations are over all $I \in \mathcal{F}(C \times D)$, $K \in \mathcal{F}(C)$ and $L \in \mathcal{F}(D)$.

Proof. Let E be the poset presented above, with universal monotone function $\gamma : \mathcal{F}(C \times D) \to E$. Because of the join stability of the relations, the binary join operation on $\mathcal{F}(C \times D)$ (i.e. the union) defines a binary operation $\vee$ on E, $\gamma(U) \vee \gamma(V) = \gamma(U \cup V)$. This is binary join for E, and in fact E is a $\vee$-semilattice with γ a homomorphism. The nullary join is $\gamma(\emptyset)$. (cf proof of Theorem 2.2.)

Now suppose $\theta : C \times D \to F$ is bilinear for some join semilattice F. The mapping $U \longmapsto \vee_{(c,d) \in U} \theta(c, d)$ respects the relations that define E, since

$$\theta(\vee_{k \in K} c_k, \vee_{l \in L} d_l) \vee \quad \vee_{(c,d) \in I} \theta(c, d)$$
$$= \quad \vee_{k \in K} \vee_{l \in L} \theta(c_k, d_l) \vee \quad \vee_{(c,d) \in I} \theta(c, d)$$

The monotone map defined by this mapping clearly commutes with the construction of join on E and so there is a (necessarily unique) join semilattice from E to F extending θ. $\square$

3. The lattice ΩW^L

If L is a poset of generators for a frame ΩX, then to examine dcpo maps $\Omega X \to \Omega W$ we are equivalently examining monotone maps $L \to \Omega W$ which satisfy certain relations R. Certainly, for any ΩX, such L and R exist since L can be taken to be ΩX and R can be taken to be the set $idl(L)$ (forget the inclusion ordering on $idl(L)$, R is discrete as a poset). The rest of the data for this standard presentation is then $\lambda \equiv \vee^{\uparrow} : R \to L$ and $\pi \equiv \{(l, r) | l \in r\} \subseteq L \times R$.

Let us first examine the set of all monotone maps $L \to \Omega W$ which will be denoted ΩW^L. In particular we will relate it to $\mathcal{U}L$, which is the set of upper closed subsets of L. $\mathcal{U}L$ is a frame since it is closed with respect to all unions and intersections.

Lemma 3.1. *(i) ΩW^L is a suplattice,*

(ii) for any $l \in L$ and $a \in \Omega W$ the map $\uparrow^a l : L \to \Omega W$ given by $\uparrow^a l(l') = \vee\{a \mid l \leq l'\}$ is monotone, i.e. $\uparrow^a l \in \Omega W^L$. Here the set $\{a \mid l \leq l'\}$ consists of the singleton $\{a\}$ if $l \leq l'$ and is empty otherwise.

(iii) For every $\phi \in \Omega W^L$

$$\phi = \bigvee\{\uparrow^a l \mid a \leq \phi(l)\}.$$

(iv) $\Omega W^L \cong \mathcal{U}L \otimes_{\mathbf{Sup}} \Omega W$.

Proof. (i) Join is calculated pointwise. So if $\phi_i \in \Omega W^L$ for some $i \in I$ an indexing set, $[\vee\phi_i](l) = \vee\{\phi_i(l) \mid i \in I\}$.

(ii) If $q_1 \leq q_2$ then $\{a \mid l \leq q_1\} \subseteq \{a \mid l \leq q_2\}$ and so $\vee\{a \mid l \leq q_1\} \leq \vee\{a \mid l \leq q_2\}$.

(iii) Since join in ΩW^L is calculated pointwise we must verify, for every $l' \in L$, that

$$\phi(l') = \bigvee\{[\uparrow^a l](l') \mid a \leq \phi(l)\}.$$

But, $[\uparrow^a l](l') = \vee\{a \mid l \leq l'\}$ and so if further $a \leq \phi(l)$, then $[\uparrow^a l](l') \leq \phi(l)$ showing that RHS≤LHS. To show LHS≤RHS, take $a = \phi(l')$ and $l = l'$ (then, certainly, $a \leq \phi(l)$ and $\phi(l') \leq [\uparrow^a l](l')$).

(iv) Define a suplattice homomorphism $\Psi : \mathcal{U}L \otimes_{\mathbf{Sup}} \Omega W \to \Omega W^L$ by $\Psi(I \otimes a) = \vee\{\uparrow^a l \mid l \in I\}$. It is easy to see that this is bilinear in each coordinate and so Ψ is well defined.

Define $\Phi : \Omega W^L \to \mathcal{U}L \otimes_{\mathbf{Sup}} \Omega W$ by

$$\Phi(\phi) = \bigvee_{l \in L} \uparrow l \otimes \phi(l).$$

Φ is a suplattice homomorphism by a routine verification given the description in (i). Now since for every upper closed subset $I \subseteq L$ (i.e. for every element of $\mathcal{U}L$), $I = \cup\{\uparrow l \mid l \in I\}$ it follows that to prove Φ and Ψ are bijections it is sufficient to show that $\Phi(\uparrow^a l) = \uparrow l \otimes a$ and $\Psi(\uparrow l \otimes a) = \uparrow^a l$ both of which are immediate. $\square$

In other words monotone maps $L \to \Omega W$ are exactly the elements of $\mathcal{U}L \otimes_{\mathbf{Sup}} \Omega W$. The next aim is to isolate which elements of $\mathcal{U}L \otimes_{\mathbf{Sup}} \Omega W$ correspond to monotone maps satisfying the relations R. Given a monotone function $\phi : L \to \Omega W$, it satisfies R if and only if its dcpo extension $\overline{\phi} : idl(L) \to \Omega W$ composes equally with the (dcpo) maps of the presentation

$$\overline{e_1}, \overline{e_2} : idl(R) \rightrightarrows idl(L).$$

This is the dcpo coequalizer form of a dcpo presentation for ΩX. But it is widely known (e.g. [V93]) that these dcpo homomorphisms (e.g. $\overline{e_1}$) correspond bijectively to frame homomorphisms $\mathcal{U}L \to \mathcal{U}R$ since:

Lemma 3.2. *For any posets L, R, (with L a join semilattice) there is a bijection between dcpo homomorphisms $\overline{h} : idl(R) \to idl(L)$ and frame homomorphism $\Omega h : \mathcal{U}L \to \mathcal{U}R$. Ωh is related to $\overline{h}$ by $\Omega h(\uparrow l) = \{r \mid l \in \overline{h}(\downarrow r)\}$.*

Proof. For any meet semilattice the set of lower closed subsets of it is well known to be the free frame on the meet semilattice, see II Thm 1.2 in [J82]. The injection of generators is $a \longmapsto \downarrow a$.

Now $idl(L)$ is equivalently the set of meet semilattice homomorphisms $L^{op} \to \Omega$, where Ω is the poset of truth values (so classically $\Omega = \{0 \le 1\}$) and L^{op} is the dual poset to L. To see this, given an ideal $I \subseteq^{\uparrow} L$ consider $\chi_I : L^{op} \to \Omega$ by $\chi_I(l) = 1$ if and only if $l \in I$.

But any dcpo homomorphism $\overline{h} : idl(R) \to idl(L)$ is equivalent to a monotone map $h : R \to idl(L)$ since $idl(R)$ is the free dcpo qua poset. By embedding $idl(L)$ in $\Omega^{L^{op}}$ and taking the exponential transpose, $h : R \to idl(L)$ corresponds to $\widetilde{h} : L^{op} \to \Omega^R$. Firstly note that $\widetilde{h}(l)$ is monotone for every $l \in L$, and so $\widetilde{h} : L^{op} \to \mathcal{U}R$. Secondly note that $\widetilde{h}$ is a meet semilattice homomorphisms by our description of $idl(L)$ in terms of meet semilattice homomorphisms. In reverse, any meet semilattice homomorphism $L^{op} \to \mathcal{U}R$ gives rise to a monotone map $R \to idl(L)$.

Now any meet semilattice homomorphism $L^{op} \to \mathcal{U}R$ extends uniquely to a frame homomorphism from the frame of lower closed subsets of L^{op} to $\mathcal{U}R$. But the set of lower closed subsets of L^{op} is the same as $\mathcal{U}L$ and so the proof is complete.

The formula given for Ωh is immediate from construction. $\square$

A monotone map $\phi : L \to \Omega W$ satisfies the equations of the presentation iff $\overline{\phi} \circ \overline{e_1} = \overline{\phi} \circ \overline{e_2}$ and so given this lemma our next objective is to show that ϕ satisfies the equations if and only if $(\Omega e_1 \otimes 1)(\Phi(\phi)) = (\Omega e_2 \otimes 1)(\Phi(\phi))$ where $\Phi : \Omega W^L \xrightarrow{\cong} \mathcal{U}L \otimes_{\mathbf{Sup}} \Omega W$ is as in the proof of Lemma 3.1 (iv) above. This would give an account of dcpo maps purely in terms of a frame equalizer. Given the description of $\Phi(\phi)$ above, we have that

$$(\Omega e_1 \otimes 1)(\Phi(\phi)) = \bigvee_{l \in L} \Omega e_1(\uparrow l) \otimes \phi(l)$$

But from the description of Ωe_1 in terms of $\overline{e_1}$ just given, we have that $\Omega e_1(\uparrow l) = \vee\{\uparrow r \mid l \in \overline{e_1}(\downarrow r)\}$. But $l \in \overline{e_1}(\downarrow r)$ implies $\downarrow l \subseteq \overline{e_1}(\downarrow r)$ and hence that $\phi(l) = \overline{\phi}(\downarrow l) \leq \overline{\phi} \circ \overline{e_1}(\downarrow r)$. Therefore

$$\bigvee_{l \in L} \Omega e_1(\uparrow l) \otimes \phi(l) \leq \bigvee_{r \in R} \uparrow r \otimes \overline{\phi} \circ \overline{e_1}(\downarrow r).$$

In the other direction, since $\overline{e_1}(\downarrow r) = \bigcup^{\uparrow}\{\downarrow l \mid l \in \overline{e_1}(\downarrow r)\}$ and $\overline{\phi}$ is a dcpo homomorphism, it is true that $\overline{\phi} \circ \overline{e_1}(\downarrow r) = \bigvee^{\uparrow}\{\phi(l) \mid l \in \overline{e_1}(\downarrow r)\}$ and so

$$\bigvee_{r \in R} \uparrow r \otimes \overline{\phi} \circ \overline{e_1}(\downarrow r) = \bigvee_{r \in R}\{\uparrow r \otimes \phi(l) \mid l \in \overline{e_1}(\downarrow r)\}$$

But $l \in \overline{e_1}(\downarrow r)$ implies $\uparrow r \leq \Omega e_1(\uparrow l)$ and so

$$(\Omega e_1 \otimes 1)(\Phi(\phi)) = \bigvee_{r \in R} \uparrow r \otimes \overline{\phi} \circ \overline{e_1}(\downarrow r) = \Phi(\overline{\phi} \circ \overline{e_1} \circ \downarrow).$$

Hence $(\Omega e_1 \otimes 1)(\Phi(\phi)) = (\Omega e_2 \otimes 1)(\Phi(\phi))$ if and only if $\overline{\phi} \circ \overline{e_1} = \overline{\phi} \circ \overline{e_2}$ since Φ is a bijection (and $\downarrow$ is universal). Therefore:

Theorem 3.3. *Given a frame ΩX and a presentation of it as a dcpo $\overline{e_1}, \overline{e_2} : idl(R) \rightrightarrows idl(L)$ then for any frame ΩW there is a bijection between the frame equalizer of*

$$(\Omega e_1 \otimes 1), (\Omega e_2 \otimes 1) : \mathcal{U}L \otimes_{\mathbf{Sup}} \Omega W \rightrightarrows \mathcal{U}R \otimes_{\mathbf{Sup}} \Omega W$$

and $\mathbf{dcpo}(\Omega X, \Omega W)$.

In particular the identity map $1_{\Omega X}$ corresponds to an element $\Phi(1_{\Omega X}) \in \mathcal{U}L \otimes_{\mathbf{Sup}} \Omega X$ since $1_{\Omega X}$ is the universal map from the lattice L to ΩX (recall that we may take $L = \Omega X$ using the standard presentation). $\Phi(1_{\Omega X})$ has the following important property.

Lemma 3.4. *For any pair of frames* ΩY, ΩX, *and any* $K \in \Omega Y \otimes_{\mathbf{Sup}} \Omega X$ *there exists a frame homomorphism* $\Omega f_K : \mathcal{U}L \to \Omega Y$ *such that*

$$K = (\Omega f_K \otimes 1)(\Phi(1_{\Omega X})).$$

Proof. Defining Ωf is equivalent to defining a meet semilattice homomorphism $L^{op} \to \Omega Y$, and this can be done by

$$q : l \longmapsto \bigvee \{b \in \Omega Y \mid b \otimes l \leq K\}.$$

Note that $q(l_1 \vee_L l_2) \geq q(l_1) \wedge_{\Omega Y} q(l_2)$ since if $b_1, b_2 \in \Omega Y$ have the property $b_i \otimes l_i \leq K$ for $i = 1, 2$ then $b_1 \wedge b_2 \otimes (l_1 \vee_L l_2) \leq K$. But

$$
\begin{aligned}
(\Omega f_K \otimes 1)(\Phi(1_{\Omega X})) &= (\Omega f_K \otimes 1) \bigvee_{l \in L} \uparrow l \otimes l \\
&= \bigvee \{b \otimes l \mid b \otimes l \leq K, l \in L\} \\
&= K
\end{aligned}
$$

where the final line follows since $L = \Omega X$ and every K is a join of suplattice generators. $\qquad\square$

4. Dcpo maps as Natural Transformations

We are now in a position to prove the main result. Recall from the introduction that for any frame ΩX we have the functors:

$$
\begin{array}{cccc}
\Lambda^{\Omega X} : \mathbf{Fr} & \to & & \mathbf{Set} \\
\Omega Y_1 & \longmapsto & \Omega Y_1 \otimes_{\mathbf{Sup}} \Omega X \\
\Omega f \downarrow & \longmapsto & \downarrow \Omega f \otimes 1_{\Omega X} \\
\Omega Y_2 & \longmapsto & \Omega Y_1 \otimes_{\mathbf{Sup}} \Omega X
\end{array}
$$

Theorem 4.1. *For any frames* ΩX *and* ΩW *there is a bijection between* $\mathbf{dcpo}(\Omega X, \Omega W)$ *and the set of natural transformations from* $\Lambda^{\Omega X}$ *to* $\Lambda^{\Omega W}$.

Proof. Assume ΩX is presented as a dcpo by L, R as in the previous section, so $1_{\Omega X} : L \to \Omega X$ is the universal map from generators (as $L = \Omega X$). Let $\alpha : \Lambda^{\Omega X} \dot{\to} \Lambda^{\Omega W}$ be a natural transformation, then $\alpha_{\mathcal{U}L}(\Phi(1_{\Omega X})) \in \mathcal{U}L \otimes_{\mathbf{Sup}} \Omega W$. But the universal map, $1_{\Omega X}$, certainly satisfies the relations R and so $(\Omega e_1 \otimes 1_{\Omega X})\Phi(1_{\Omega X}) = (\Omega e_1 \otimes 1_{\Omega X})\Phi(1_{\Omega X})$ by Theorem 3.3 and so

$$(\Omega e_1 \otimes 1_{\Omega W})\alpha_{\Omega X}(\Phi(1_{\Omega X})) = (\Omega e_1 \otimes 1_{\Omega W})\alpha_{\Omega X}(\Phi(1_{\Omega X}))$$

by naturality of α. Hence, also by Theorem 3.3 (in the opposite direction), $\alpha_{\Omega X}(\Phi(1_{\Omega X}))$ corresponds to a dcpo homomorphism, which we denote q^α.

In the other direction, given $q : \Omega X \to \Omega W$, to define $\alpha^q_{\Omega Y} :$ $\Omega Y \otimes_{\mathbf{Sup}} \Omega X \to \Omega Y \otimes_{\mathbf{Sup}} \Omega W$ for every ΩY, it is sufficient by Proposition 2.4 to define $\Omega Y \otimes_{\vee \mathbf{SLat}} \Omega X \to \Omega Y \otimes_{\mathbf{Sup}} \Omega W$ which satisfy the equation $R^\otimes_{\mathbf{dcpo}}$. Consider, for any frame ΩY, the assignments $q_{\Omega Y}$ given by

$$
\begin{aligned}
\Omega Y \otimes_{\vee \mathbf{SLat}} \Omega X \quad &\to \quad \Omega Y \otimes_{\mathbf{Sup}} \Omega W \\
\vee_{i \in I} b_i \otimes a_i \quad &\longmapsto \quad \bigvee_{I' \in \mathcal{F}I} \wedge_{i \in I'} b_i \otimes q(\vee_{i \in I'} a_i)
\end{aligned}
$$

where the join is taken over all the finite subsets of the (finite) indexing set I. It must be checked that these are well defined poset maps, and this is going to be done by appealing to the description of join semilattice tensor as given in Proposition 2.5. Firstly note that if we add a single element to an indexing set, i.e. $J = I \sqcup \{*\}$, then $\mathcal{F}J = \mathcal{F}I \amalg \mathcal{F}I$ since every subset of J contains $*$ or it doesn't. It follows that for any pair $(c, d) \in \Omega Y \times \Omega X$

$$
\begin{aligned}
&q_{\Omega Y}(c \otimes d \vee [\vee_{i \in I} b_i \otimes a_i]) \\
= \quad &q_{\Omega Y}(\vee_{i \in I} b_i \otimes a_i) \vee \bigvee_{I' \in \mathcal{F}I} [c \wedge \wedge_{i \in I'} b_i] \otimes q_{\Omega Y}(d \vee \vee_{i \in I'} a_i) \quad \text{(a)}
\end{aligned}
$$

Take $c = \vee_K c_k$, $d = \vee_L d_l$ as in the statement of Proposition 2.5. Then

$$
\begin{aligned}
&q_{\Omega Y}([\vee_{K \times L} c_k \otimes d_l] \vee [\vee_{i \in I} b_i \otimes a_i]) \\
= \quad &\bigvee_{I' \in \mathcal{F}I} \bigvee_{J' \in \mathcal{F}(K \times L)} \chi_{I', J'} \quad \text{(b)}
\end{aligned}
$$

where $\chi_{I', J'} = [(\wedge_{(k,l) \in J'} c_k) \wedge (\wedge_{i \in I'} b_i)] \otimes q_{\Omega Y}[(\vee_{(k,l) \in J'} d_l) \vee (\vee_{i \in I'} a_i)]$ and we must verify that (a)=(b) in order to establish that each $q_{\Omega Y}$ is a well defined monotone map. Firstly (a) $\leq$ (b) since by taking $J' = \emptyset$ it is clear that $q_{\Omega Y}(\vee_{i \in I} b_i \otimes a_i) \leq$ (b) and then for any $k \in K$, $[c_k \wedge (\wedge_{i \in I'} b_i)] \otimes q_{\Omega Y}[(\vee_{l \in L} d_l) \vee (\vee_{i \in I'} a_i)] \leq$ (b) by taking $J' = \{k\} \times L$ in (b).

To prove that (b)$\leq$(a) note that for J' empty it needs to be shown that $q_{\Omega Y}(\vee_{i \in I} b_i \otimes a_i) \leq$ (a), which is trivial. On the other

hand given $(k', l') \in J'$ we have that

$$[(\wedge_{(k,l) \in J'} c_k) \wedge (\wedge_{i \in I'} b_i)] \otimes q_{\Omega Y}[(\vee_{(k,l) \in J'} d_l) \vee (\vee_{i \in I'} a_i)]$$
$$\leq \quad [c_{k'} \wedge (\wedge_{i \in I'} b_i)] \otimes q_{\Omega Y}[(\vee_{(k,l) \in J'} d_l) \vee (\vee_{i \in I'} a_i)]$$
$$\leq \quad [c \wedge (\wedge_{i \in I'} b_i)] \otimes q_{\Omega Y}(d \vee \vee_{i \in I'} a_i)$$

and so (a)=(b) and $q_{\Omega Y}$ is well defined.

To show that further $q_{\Omega Y}$ determines a map $\Omega Y \otimes_{\mathbf{Sup}} \Omega X \to \Omega Y \otimes_{\mathbf{Sup}} \Omega W$ it must be verified that it satisfies the equations $R^{\otimes}_{\mathbf{dcpo}}$. Let $u = \vee_{i \in I} b_i \otimes a_i$ denote a typical element of $\Omega Y \otimes_{\vee \mathbf{SLat}} \Omega X$, as used in the equations $R^{\otimes}_{\mathbf{dcpo}}$ of Proposition 2.4. Then, repeating (a) above, we have that

$$q_{\Omega Y}((\_) \otimes (\_) \vee u)$$
$$= \quad q_{\Omega Y}(\vee_{i \in I} b_i \otimes a_i) \vee \bigvee_{I' \in \mathcal{F}I} [(\_) \wedge \wedge_{i \in I'} b_i] \otimes q((\_) \vee \vee_{i \in I'} a_i)$$

which is easily seen to be dcpo bilinear since directed join commutes with $\otimes$, finite join and finite meet. $q_{\Omega Y}$ therefore determines a dcpo homomorphism

$$\alpha^q_{\Omega Y} : \Omega Y \otimes_{\mathbf{Sup}} \Omega X \to \Omega Y \otimes_{\mathbf{Sup}} \Omega W$$

for every frame ΩY. Checking naturality of α^q is easy since given any $\Omega f : \Omega Y_1 \to \Omega Y_2$,

$$[\Omega f \otimes 1_{\Omega W}] \bigvee_{I' \in \mathcal{F}I} \wedge_{i \in I'} b_i \otimes q(\vee_{i \in I'} a_i)$$
$$= \quad \bigvee_{I' \in \mathcal{F}I} \wedge_{i \in I'} \Omega f(b_i) \otimes q(\vee_{i \in I'} a_i)$$
$$= \quad \alpha^q_{\Omega Y_2}([\Omega f \otimes 1_{\Omega X}] \vee_{i \in I} b_i \otimes a_i)$$

for any dcpo generator $\vee_{i \in I} b_i \otimes a_i$ of $\Omega Y_1 \otimes_{\mathbf{Sup}} \Omega X$. (Note that, of course, the composite $\Omega X \times \Omega Y \xrightarrow{\otimes} \Omega Y \otimes_{\vee \mathbf{SLat}} \Omega X \to \Omega Y \otimes_{\mathbf{Sup}} \Omega X$, where the second map is the universal map given by the presentation $R^{\otimes}_{\mathbf{dcpo}}$, is just the universal suplattice tensor map, and so we have not overburdened notation by introducing two separate tensors on elements.)

To complete the proof it must be checked that the constructions are in bijection. This is eased considerably by the observation that for any $\alpha : \Lambda^{\Omega X} \dot{\to} \Lambda^{\Omega W}$, α is uniquely determined by $\alpha_{\Omega X}(\Phi(1_{\Omega X}))$. This can be seen by applying Lemma 3.4 since

this shows that for any frame ΩY and any $K \in \Omega Y \otimes_{\mathbf{Sup}} \Omega X$ we have $\alpha_{\Omega Y}(K) = \alpha_{\Omega Y}([\Omega f_K \otimes 1]\Phi(1_{\Omega X})) = [\Omega f_K \otimes 1]\alpha_{\Omega X}(\Phi(1_{\Omega X}))$. Therefore the class of natural transformations injects into the set of dcpo maps $\mathbf{dcpo}(\Omega X, \Omega W)$. Finally therefore, it must be checked that given a dcpo map $q : \Omega X \to \Omega W$ that

$$\alpha^q_{\mathcal{U}L}(\Phi(1_{\Omega X})) = \Phi(q)$$

where there is no need to distinguishing between the poset map $q : L \to \Omega W$ and the dcpo map $q : \Omega X \to \Omega W$ since the universal map can be taken to be the identity. Now $\Phi(1_{\Omega X}) = \bigvee_{l \in L} \uparrow l \otimes l$. For every finite subset L' of L we have that

$$\alpha^q_{\mathcal{U}L}(\vee_{l \in L'} \uparrow l \otimes l) = \bigvee_{I' \in \mathcal{F}L'} \wedge_{l \in I'} \uparrow l \otimes q(\vee_{l \in I'} l)$$
$$= \bigvee_{I' \in \mathcal{F}L'} \uparrow \vee I' \otimes q(\vee I'),$$

where the last line is by the definition of finite join. Therefore $\alpha^q_{\mathcal{U}L}(\Phi(1_{\Omega X})) = \bigvee_{l \in L} \uparrow l \otimes q(l) = \Phi(q)$.   □

This is the main result. The dcpo homomorphisms can be represented as natural transformations, providing a new category in which to establish results about them.

We did not use this, but it will be useful to observe for later on that, $\alpha_\Omega = q^\alpha$ where Ω is the set of truth values (trivially a frame since, for example, $\Omega = P(\{*\})$, i.e. the power set on the singleton set). Now $\alpha_\Omega : \Omega \otimes_{\mathbf{Sup}} \Omega X \to \Omega \otimes_{\mathbf{Sup}} \Omega W$, but $\Omega \otimes_{\mathbf{Sup}} \Omega X \cong \Omega X$ since $P(\{*\})$ is the free suplattice on $\{*\}$, and so the statement $\alpha_\Omega = q^\alpha$ is correctly typed (up to isomorphism). Now $\Phi(q^\alpha) = \bigvee_{l \in L} \uparrow l \otimes q^\alpha(l)$, and by Lemma 3.4 for every $a \in \Omega X$, there exists $\Omega p_a : \mathcal{U}L \to \Omega$ (defined by $\Omega p_a(\uparrow l) = \bigvee\{1 \in \Omega \mid l \le a\}$) such that $(\Omega p_a \otimes 1)(\Phi(1_{\Omega X})) = 1 \otimes a$ (i.e. the image of a under $\Omega \otimes_{\mathbf{Sup}} \Omega X \cong \Omega X$). Hence by naturality

$$\begin{aligned}\alpha_\Omega(a) &= \bigvee_{l \in L} \Omega p_a \uparrow l \otimes q^\alpha(l) \\ &= q^\alpha(a).\end{aligned}$$

Using this we have that,

Lemma 4.2. *the bijection of the theorem preserves function composition in both directions.*

Proof. Since it is a bijection, it is sufficient to verify that it preserves function composition one way round only. It has just been established that the bijection, one way round, is $\alpha \longmapsto \alpha_\Omega$ and so this is immediate since natural transformation composition is calculated componentwise. $\square$

4.1. **Distributive lattices internal to $[\mathbf{Fr}, \mathbf{Set}]$.** The category of functors $[\mathbf{Fr}, \mathbf{Set}]$ (with morphisms all natural transformations) is well behaved in that it has finite products. The nullary product (i.e. the terminal object) is the functor that sends every frame to $\{*\}$ and every morphisms to the identity morphism. Given two functors $F_1, F_2 : \mathbf{Fr} \to \mathbf{Set}$ their binary product $(F_1 \times F_2)$ is given by

$$
\begin{aligned}
[F_1 \times F_2](\Omega Y) &= F_1(\Omega Y) \times_{\mathbf{Set}} F_2(\Omega Y) \\
[F_1 \times F_2](\Omega f &: \Omega Y_1 \to \Omega Y_2) = F_1(\Omega f) \times F_1(\Omega f).
\end{aligned}
$$

It is easy to verify that this construction gives the correct universal characteristic of a categorical product. In any category with finite product the notion of an internal distributive lattice makes sense since all the equations of such a theory can be written out as assertions that certain diagrams commute, and the only constructions involved in making these diagrams are finite products.

Proposition 4.3. *For any frame ΩX, $\Lambda^{\Omega X}$ is an internal distributive lattice in $[\mathbf{Fr}, \mathbf{Set}]$.*

Proof. We define join only, leaving the further routine verifications to the reader.

We need to define a natural transformation $\sqcup : \Lambda^{\Omega X} \times \Lambda^{\Omega X} \,\dot\to\, \Lambda^{\Omega X}$. For every ΩY define $\sqcup_{\Omega Y} : \Omega Y \otimes_{\mathbf{Sup}} \Omega X \times \Omega Y \otimes_{\mathbf{Sup}} \Omega X \to \Omega Y \otimes_{\mathbf{Sup}} \Omega X$ to be the join operation on the suplattice $\Omega Y \otimes_{\mathbf{Sup}} \Omega X$. It is trivially natural since for any $\Omega f : \Omega Y_1 \to \Omega Y_2$, $\Omega f \otimes 1 : \Omega Y_1 \otimes_{\mathbf{Sup}} \Omega X \to \Omega Y_2 \otimes_{\mathbf{Sup}} \Omega X$ is a join semilattice homomorphism. $\square$

Given the simplicity of this construction the characterization of those natural transformation that are internal meet or join semilattice homomorphism is straightforward:

Lemma 4.4. *A natural transformation* $\alpha : \Lambda^{\Omega X} \to \Lambda^{\Omega W}$ *is an internal meet/join semilattice homomorphism iff* $\alpha_{\Omega Y} : \Omega Y \otimes_{\mathbf{Sup}} \Omega X \to \Omega Y \otimes_{\mathbf{Sup}} \Omega W$ *is a meet/join lattice homomorphism for every* ΩY.

Proof. This is immediate from construction given that a commutative square of natural transformations commutes if and only if they commute when evaluated at each object. $\square$

4.2. **Frame homomorphisms as Natural Transformations.** We are now in a position to state a corollary to the main theorem, showing now to represent frame homomorphisms in terms of natural transformations.

Theorem 4.5. *(a) There is a bijection between suplattice homomorphisms* $\Omega X \to \Omega W$ *and internal join semilattice homomorphisms* $\Lambda^{\Omega X} \dot\to \Lambda^{\Omega W}$.

(b) There is a bijection between frame homomorphisms $\Omega X \to \Omega W$ *and internal distributive lattice homomorphisms* $\Lambda^{\Omega X} \dot\to \Lambda^{\Omega W}$.

Proof. (a) Firstly if $\alpha : \Lambda^{\Omega X} \dot\to \Lambda^{\Omega W}$ is an internal join semilattice homomorphism then by the last lemma $\alpha_{\Omega Y}$ is a join semilattice homomorphism for every ΩY, but this covers the case $\Omega Y = \Omega$, and so α_Ω is a join semilattice homomorphism. But by the comments after the main theorem it is shown that $\alpha_\Omega = q^\alpha$ and so the corresponding dcpo homomorphism is also a join semilattice homomorphism, i.e. a suplattice homomorphism.

Conversely say $q : \Omega X \to \Omega W$ is a suplattice homomorphism, then for any frame ΩY, the map $\alpha^q_{\Omega Y}$

$$\begin{array}{ccc} \Omega Y \otimes_{\mathbf{Sup}} \Omega X & \to & \Omega Y \otimes_{\mathbf{Sup}} \Omega W \\ \vee_{i \in I} b_i \otimes a_i & \longmapsto & \bigvee_{I' \in \mathcal{F}I} \wedge_{i \in I'} b_i \otimes q(\vee_{i \in I'} a_i) \end{array}$$

reduces to

$$\vee_{i \in I} b_i \otimes a_i \longmapsto \vee_{i \in I} b_i \otimes q(a_i)$$

which preserves finite joins. Since $\alpha^q_{\Omega Y}$ preserves finite joins for every ΩY this is sufficient by the last lemma to show that α^q is an internal join semilattice homomorphism.

(b) Certainly if $\alpha : \Lambda^{\Omega X} \dot\to \Lambda^{\Omega W}$ is an internal distributive lattice homomorphism then, again using the last lemma, α_Ω is a distributive lattice homomorphism, and so $q^\alpha = \alpha_\Omega$ is a frame homomorphism.

In the other direction, if $q : \Omega X \to \Omega W$ is a frame homomorphism, then it is certainly a suplattice homomorphism and so, as in (a) $\alpha^q_{\Omega Y} = 1 \otimes q$. But suplattice tensor is frame coproduct and so $\alpha^q_{\Omega Y}$ is a frame homomorphism for each ΩY implying that α^q is an internal distributive lattice homomorphism as required.   □

Thus the representation theorem for dcpo homomorphisms between frames specializes naturally to frame homomorphisms, and so locale maps. This is therefore a representation theorem for a well understood notion of continuity, i.e. continuity as defined by locale theory.

4.3. **Preframes.** In this subsection we digress briefly to verify the obvious missing part to the previous theorem:

Theorem 4.6. *There is a bijection between preframe homomorphisms $\Omega X \to \Omega W$ and internal meet semilattice homomorphisms $\Lambda^{\Omega X} \dot\to \Lambda^{\Omega W}$.*

A preframe is a dcpo which is also a meet semilattice and enjoys the distributivity property

$$a \wedge \bigvee^{\uparrow} T = \bigvee^{\uparrow} \{a \wedge t \mid t \in T\}$$

for any directed subset T and element a. Preframe homomorphism preserve directed joins and finite meets and so a category **PFr** is defined. The theory of preframes lies between the theory of dcpos and the theory of frames in much the same way that suplattice theory does. See e.g. Ch. 3 of [T96] for a summary of the relationship between suplattices and preframes. The difference is that the finitary data for preframes (finite meets) is dual to the finitary data for suplattices (finite joins). Preframe presentations do present ([JV91]) and therefore a preframe tensor can be defined. To prove that preframe presentations present, re-apply Lemma 2.1 and Theorem 2.2, but with frames in the place of suplattices and preframes in the place of dcpos. The constructions are identical with the free frame qua preframe being again the set of Scott closed subsets, and

preframe image factorization again being constructed as the intersection of all subpreframes containing a set-theoretic image. The existence of frame coequalizers needed for the re-application is well known, for example II 2.11 of [J82]. Just as the suplattice tensor corresponds to frame coproduct so too does preframe tensor (once the existence of a preframe tensor is established, that it defines frame coproduct is a routine verification). $a \odot b \in \Omega X \otimes_{\mathbf{PFr}} \Omega Y$ is notation for a typical generator of a preframe tensor and Ω is the preframe tensor unit (so $\Omega X \otimes_{\mathbf{PFr}} \Omega \cong \Omega X$ for any ΩX).

We can now prove the theorem.

Proof. If $\alpha : \Lambda^{\Omega X} \dot\to \Lambda^{\Omega W}$ is an internal meet semilattice homomorphism, then α_Ω is a meet semilattice homomorphism and so q^α is a meet semilattice homomorphism and a dcpo homomorphism, i.e. a preframe homomorphism.

Conversely, given $q : \Omega X \to \Omega W$, a preframe homomorphism, then define for any frame ΩY the map $\overline{\alpha}^q_{\Omega Y}$ by

$$\begin{aligned} \Omega Y \otimes_{\mathbf{PFr}} \Omega X &\quad\to\quad \Omega Y \otimes_{\mathbf{PFr}} \Omega W \\ \wedge_{i \in I} b_i \odot a_i &\quad\longmapsto\quad \wedge_{i \in I} b_i \odot q(a_i), \end{aligned}$$

i.e. $\overline{\alpha}^q_{\Omega Y} = 1_{\Omega Y} \otimes q$ where $\otimes$ is preframe tensor. This defines a natural transformation $\overline{\alpha}^q : \Lambda^{\Omega X} \dot\to \Lambda^{\Omega W}$ since preframe tensor is frame coproduct (if $\Omega f : \Omega Y_1 \to \Omega Y_2$, then $[\Omega f +_{\mathbf{Fr}} 1_{\Omega X}](b \odot a) = \Omega f(b) \odot a$). $\overline{\alpha}^q$ is an internal meet semilattice homomorphism since every $\overline{\alpha}^q_{\Omega Y}$ is a meet semilattice homomorphism.

But, by the remarks after the proof of the main theorem, we have that for any $\alpha, \beta : \Lambda^{\Omega X} \dot\to \Lambda^{\Omega W}$, to prove that $\alpha = \beta$ it is sufficient to prove that $\alpha_\Omega = \beta_\Omega$. But $\overline{\alpha}^q_\Omega = q = \alpha^q_\Omega$ and so α^q is an internal meet semilattice homomorphism. $\square$

5. Locale Theoretic interpretation

Our next aim is to restate the results in terms of locale theory, as it is with this interpretation that we can see that the functors $\Lambda^{\Omega X}$ can in fact be understood to correspond to function spaces in a natural way.

The category of locales, **Loc**, is defined as the opposite of the category of frames. It is used extensively to investigate topology in a constructive context since, for example, a Tychonoff theorem is

true of locales without an assumption of the axiom of choice (e.g. [JV91]). Our notation has reflected locale theory since for every frame ΩX there is a corresponding locale X and for every frame homomorphism $\Omega f : \Omega Y \to \Omega X$ there is a corresponding locale map $f : X \to Y$. Locale maps $f : X \to Y$ are the localic models for continuous maps between topological spaces.

The set of truth values Ω is the free suplattice on $1 = \{*\}$, and from this fact it is clear that it is the initial frame, i.e. $\Omega = \Omega 1$ where 1 is the terminal locale. Given a locale X the *set of points* of X is the set of locale maps $\mathbf{Loc}(1, X) \equiv \mathbf{Fr}(\Omega X, \Omega)$. The points of a locale can be given a topology in an obvious way, and this defines a functor $pt : \mathbf{Loc} \to \mathbf{Top}$, to the category of topological spaces. Given any locale X we refer to ΩX as the corresponding frame of *opens*.

The purpose of this section is to restate our results for the category of locales.

Definition 5.1. For any poset L define the locale $Idl(L)$ by

$$\Omega Idl(L) = \mathcal{U}L.$$

The set of points of $Idl(L)$ are exactly the frame homomorphism $\mathcal{U}L \to \Omega$, but $\Omega = \mathcal{U}1$ and so by 3.2, the points are exactly dcpo homomorphisms $1 \to idl(L)$, i.e. exactly the ideals of the poset L. This is why the notation $Idl(L)$ is used. By looking at the poset $\mathbf{2} = \{0 \le 1\}$, we define the Sierpiński locale $\mathbb{S} = Idl(\mathbf{2})$. The Sierpiński locale therefore has Ω as its set of points, and so, classically, corresponds to a two point topological space. The opens of the Sierpiński are the upward closed subsets of $\mathbf{2}$, and so, classically, $\Omega\mathbb{S} = \mathbf{3}$, corresponding to the usual definition of the Sierpiński topological space. (By 'classically' we mean under the assumption $\Omega = \mathbf{2}$.) The Sierpiński locale enjoys the following key property:

Proposition 5.2. *For any locale X, $\mathbf{Loc}(X, \mathbb{S}) \cong \Omega X$.*

Proof. From the proof of Lemma 3.2, for any lattice L, frame homomorphisms $\mathcal{U}L \to \mathbf{\Omega}X$ are exactly meet semilattice homomorphism $L^{op} \to \mathbf{\Omega}X$ since $\mathcal{U}L$ is the set of lower closed subsets of L^{op}. Here $L = \mathbf{2}$ and so this result follows by verifying that $\mathbf{\Omega}X$ is in bijection with meet semilattice homomorphisms $\mathbf{2}^{op} \to \Omega X$. $\qquad\square$

Therefore define

$$\begin{array}{rcc}
\mathbb{S}^X : \mathbf{Loc}^{op} & \to & \mathbf{Set} \\
Y_1 & \longmapsto & \mathbf{Loc}(Y_1 \times X, \mathbb{S}) \\
f \uparrow & \longmapsto & \downarrow \mathbf{Loc}(f \times 1_X, 1_{\mathbb{S}}) \\
Y_2 & \longmapsto & \mathbf{Loc}(Y_2 \times X, \mathbb{S})
\end{array}$$

where $\mathbf{Loc}(f \times 1_X, 1)(Y_1 \times X \overset{a}{\to} \mathbb{S}) = Y_2 \times X \overset{f \times 1_X}{\to} Y_1 \times X \overset{a}{\to} \mathbb{S}$, i.e. function composition, and so since locale product is the same as frame coproduct, given the proposition we have that:

Theorem 5.3. *For any locale X, $\Lambda^{\Omega X}$ is naturally isomorphic to $\mathbb{S}^X$.*

The notation $\mathbb{S}^X$ is suggestive of an exponential, and so the next task is to prove that the functor is indeed an internal exponential in the functor category $[\mathbf{Loc}^{op}, \mathbf{Set}]$.

Theorem 5.4. $\mathbb{S}^X$ *is the exponential $\mathbf{Loc}(\_, \mathbb{S})^{\mathbf{Loc}(\_, X)}$ in $[\mathbf{Loc}^{op}, \mathbf{Set}]$, where for any locale X, $\mathbf{Loc}(\_, X) : \mathbf{Loc}^{op} \to \mathbf{Set}$, takes Y to $\mathbf{Loc}(Y, X)$ and locale maps to function composition in $\mathbf{Set}$ (i.e. $\mathbf{Loc}(\_, X)$ is the image of the Yoneda embedding).*

Proof. It is well known from category theory (for example, use the proof of A1.5.5 in [J02]) that in any functor category $[\mathcal{C}^{op}, \mathbf{Set}]$ with objects G, F the exponential G^F is well defined if the class of natural transformations, $Nat[\mathcal{C}(\_, A) \times F, G]$, is a set for every object A in $\mathcal{C}$. By Yoneda's lemma (i.e. the assertion $Nat[\mathcal{C}(\_, A), F] \cong F(A)$ for any F, A) $Nat[\mathbf{Loc}(\_, Y) \times \mathbf{Loc}(\_, X), \mathbf{Loc}(\_, \mathbb{S})]$ is a set since $\mathbf{Loc}(\_, Y) \times \mathbf{Loc}(\_, X) \cong \mathbf{Loc}(\_, Y \times X)$ and so $\mathbb{S}^X$ exists. Also by Yoneda's lemma, note that if G^F is defined then it must be given by $G^F(A) = Nat[\mathcal{C}(\_, A) \times F, G]$. $\square$

Heading towards a statement of the main theorem in terms of locale theory, a localic interpretation for dcpo homomorphisms is useful. Given a locale X certainly ΩX is a dcpo, and so we can define a new locale $\mathbb{P}X$ by

$$\Omega \mathbb{P}X = \mathbf{Fr}\langle \Omega X \ qua \ dcpo \rangle$$

since we have already discussed that frame coequalizers exist and so frame presentations are well defined. $\mathbb{P}X$ is known as the double power locale on X and its study is advocated in [V93].

(The study of power locales is broadly related to the topological study of hyperspaces.) From this definition we have that for any locale W

$$\mathbf{Loc}(W, \mathbb{P}X) \cong \mathbf{dcpo}(\Omega X, \Omega W)$$

and indeed from the uniqueness part of the universal definition of presentation, it is clear that this bijection is natural in W.

Theorem 5.5. *For any locale X there are bijections, both natural in W,*
   (i) between $\mathbf{Loc}(W, \mathbb{P}X)$ and natural transformations $\mathbb{S}^X \dot\to \mathbb{S}^W$
and
   (ii) between $\mathbf{Loc}(W, X)$ and internal distributive lattice homomorphisms $\mathbb{S}^X \dot\to \mathbb{S}^W$.

Proof. This is just a re-statement of the main theorem. Since the bijection of the main theorem preserves composition (Lemma 4.2) naturality is immediate.   □

Now, for any locale X it is known that X is exponentiable if and only if it is locally compact, and certainly it is not true that all locales are locally compact. In fact VII 4.10 of [J82] shows that a locale X is exponentiable if and only if $\mathbb{S}^X$ exists as a locale (we only have it as a functor). Remarkably, by embedding $\mathbf{Loc}$ into the category $[\mathbf{Loc}^{op}, \mathbf{Set}]$ (via $X \longmapsto \mathbf{Loc}(\_, X)$) we have that $\mathbb{S}^{\mathbb{S}^X}$ exists as a locale *for every* X.

Corollary 5.6. *If X is a locale then the exponential $\mathbb{S}^{\mathbb{S}^X}$ exists in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ and is naturally isomorphic to the functor $\mathbf{Loc}(\_, \mathbb{P}X)$.*

Proof. For $\mathbb{S}^{\mathbb{S}^X}$ to exist we need only verify that the class of natural transformations $\mathbf{Loc}(\_, W) \times \mathbb{S}^X \to \mathbf{Loc}(\_, \mathbb{S})$ exists as a set. But dcpo homomorphisms $\Omega X \to \Omega W$ have been characterized, naturally in W, as the natural transformations $\mathbb{S}^X \dot\to \mathbb{S}^W$, i.e. exactly the natural transformations $\mathbf{Loc}(\_, W) \times \mathbb{S}^X \to \mathbf{Loc}(\_, \mathbb{S})$ by the definition of the exponential $\mathbb{S}^W$ in $[\mathbf{Loc}^{op}, \mathbf{Set}]$. Given this the exponential $\mathbb{S}^{\mathbb{S}^X}$ is defined by

$$\mathbb{S}^{\mathbb{S}^X}(W) = Nat[\mathbf{Loc}(\_, W) \times \mathbb{S}^X, \mathbf{Loc}(\_, \mathbb{S})]$$

and so $\mathbb{S}^{\mathbb{S}^X}(W) \cong \mathbf{Loc}(W, \mathbb{P}X)$ naturally in W.   □

6. Topos Theoretic interpretation

Care has been taken to argue constructively throughout. No use has been made of the excluded middle or any choice principals. In the proof of the main theorem, we did argue by cases, but only in a context (that of finite subsets of finite sets) where membership is decidable (see e.g. D5.4 of [J02]). It was because the arguments were constructive that Ω (the set of subsets of $1 = \{*\}$) was distinguished from $\mathbf{2} = \{0{\leq}1\}$, i.e. the poset of two elements found by taking the coproduct of 1 and 1. Further, no use has been made of a natural numbers object. It follows that the results are true relative to any elementary topos $\mathcal{E}$. In other words we could have replaced $\mathbf{Set}$ with an arbitrary topos $\mathcal{E}$ in all the work above and shown that dcpo homomorphisms are equivalent to natural transformations between certain functors in $[\mathbf{Loc}_{\mathcal{E}}^{op},\mathcal{E}]$.

We now look at the situation where $\mathcal{E}$ is a locally small topos. In other words, for any objects B, A in $\mathcal{E}, \mathcal{E}(B, A)$ is required to be a set. There is then a functor $\gamma_* : \mathcal{E} \to \mathbf{Set}$ given by $\gamma_*(A) = \mathcal{E}(1, A)$, that is $\gamma_*(A)$ is the set of points of A. Now for every topological space X in $\mathbf{Set}$, the topos of sheaves over X, denoted $Sh(X)$ is locally small. It is well known then that $\gamma_*(\Omega_{Sh(X)}) = \Omega X$, the opens of X, where $\Omega_{Sh(X)}$ is the object of truth values in the topos $Sh(X)$. Internally to $Sh(X)$ the object of truth values is always discrete and compact Hausdorff as a topology since it is the power set of $\{*\}$; but ΩX is the set of opens of an arbitrary topological space. Thus there is a significant difference between the external structure of objects (i.e. the structure of $\gamma_*(A)$) and their internal structure. Consult [J02] for background on toposes. In contrast to Proposition 5.2,

Proposition 6.1. *For any locale X in a locally small topos $\mathcal{E}$,* $\mathbf{Loc}_{\mathcal{E}}(X, \mathbb{S}) \cong \gamma_*(\Omega_{\mathcal{E}} X)$ *where $\mathbb{S}$ is the Sierpiński locale in $\mathcal{E}$.*

Proof. Proposition 5.2 carried out internally in $\mathcal{E}$ proves that $\Omega_{\mathcal{E}} X \cong F^{X,\mathbb{S}}$ where $F^{X,\mathbb{S}}$ is the object of frame homomorphism from $\Omega_{\mathcal{E}}\mathbb{S}$ to $\Omega_{\mathcal{E}} X$ internal to $\mathcal{E}$. Such an object can be constructed explicitly in any topos, e.g. as a subobject of $\Omega_{\mathcal{E}} X^{\Omega_{\mathcal{E}}\mathbb{S}}$. Therefore $\gamma_*\Omega_{\mathcal{E}} X \cong \gamma_* F^{X,\mathbb{S}}$ but $\gamma_* F^{X,\mathbb{S}} \cong \mathbf{Loc}_{\mathcal{E}}(X, \mathbb{S})$ by construction of $F^{X,\mathbb{S}}$. $\square$

What we now show is that given such $\gamma_* : \mathcal{E} \to \mathbf{Set}$, the external set $\mathbf{dcpo}_{\mathcal{E}}(\Omega_{\mathcal{E}} X, \Omega_{\mathcal{E}} W)$ is also equivalent to a set of natural transformations, now in $[\mathbf{Loc}_{\mathcal{E}}^{op}, \mathbf{Set}]$, for any internal frames $\Omega_{\mathcal{E}} X$ and $\Omega_{\mathcal{E}} W$ of $\mathcal{E}$.

To define the relevant functors note that given any $F : \mathbf{Loc}_{\mathcal{E}}^{op} \to \mathcal{E}$, there is a functor $\gamma_* \circ F : \mathbf{Loc}_{\mathcal{E}}^{op} \to \mathbf{Set}$, and for any natural transformation $\alpha : F \xrightarrow{\cdot} G$ between such functors there is a natural transformation $\gamma_* \alpha : \gamma_* F \xrightarrow{\cdot} \gamma_* G$ defined by $(\gamma_* \alpha)_Y = \gamma_*(\alpha_Y)$.

Lemma 6.2. *For any locale X in $\mathbf{Loc}_{\mathcal{E}}$*

$$\mathbf{Loc}_{\mathcal{E}}(\_, \mathbb{S})^{\mathbf{Loc}_{\mathcal{E}}(\_, X)} \cong \gamma_* \Lambda^{\Omega_{\mathcal{E}} X}$$

in $[\mathbf{Loc}_{\mathcal{E}}^{op}, \mathbf{Set}]$.

Proof. For any frame $\Omega_{\mathcal{E}} Y$ in $\mathbf{Fr}_{\mathcal{E}}$, $\gamma_* \Lambda^{\Omega_{\mathcal{E}} X}(\Omega_{\mathcal{E}} Y) = \gamma_*(\Omega_{\mathcal{E}} Y +_{\mathbf{Fr}} \Omega_{\mathcal{E}} X) = \mathbf{Loc}_{\mathcal{E}}(Y \times X, \mathbb{S})$. But $\mathbf{Loc}_{\mathcal{E}}(\_ \times X, \mathbb{S}) \cong \mathbf{Loc}_{\mathcal{E}}(\_, \mathbb{S})^{\mathbf{Loc}_{\mathcal{E}}(\_, X)}$ just as in Theorem 5.4 above. $\square$

Given this, for any such X we shall now use $\mathbb{S}^X$ to denote the functor $\mathbf{Loc}_{\mathcal{E}}(\_, \mathbb{S})^{\mathbf{Loc}_{\mathcal{E}}(\_, X)} : \mathbf{Loc}_{\mathcal{E}}^{op} \to \mathbf{Set}$ and $\Lambda^{\Omega_{\mathcal{E}} X}$ for the object of $[\mathbf{Loc}_{\mathcal{E}}^{op}, \mathcal{E}]$. We can now state our final theorem, which is a reformulation of the main result, but giving a description of the set of internal dcpo homomorphisms in a topos in terms of an external collection of natural transformations.

Theorem 6.3. *For any locale X in a locally small topos $\mathcal{E}$ there are bijections (natural in locales W)*
(a) between $\mathbf{dcpo}_{\mathcal{E}}(\Omega_{\mathcal{E}} X, \Omega_{\mathcal{E}} W)$ and $\mathbb{S}^X \xrightarrow{\cdot} \mathbb{S}^W$, and
(b) between $\mathbf{Loc}_{\mathcal{E}}(W, X)$ and internal distributive lattice homomorphisms $\mathbb{S}^X \xrightarrow{\cdot} \mathbb{S}^W$ in $[\mathbf{Loc}_{\mathcal{E}}^{op}, \mathbf{Set}]$.

Proof. (a) $\mathbf{dcpo}_{\mathcal{E}}(\Omega_{\mathcal{E}} X, \Omega_{\mathcal{E}} W) \cong \mathcal{E}(1, E^{W,X})$ where the object $E^{W,X}$ is the frame equalizer defined in Theorem 3.3. But the results above have shown $E^{W,X} \cong Nat[\Lambda^{\Omega_{\mathcal{E}} X}, \Lambda^{\Omega_{\mathcal{E}} W}]$ naturally in W. So it remains to check that $\mathcal{E}(1, Nat[\Lambda^{\Omega_{\mathcal{E}} X}, \Lambda^{\Omega_{\mathcal{E}} W}]) \cong Nat[\mathbb{S}^X, \mathbb{S}^W]$ naturally in W.

Recall that $\Phi(1_{\Omega_{\mathcal{E}} X}) \in \mathcal{E}(1, \mathcal{U}L \otimes_{\mathbf{Sup}_{\mathcal{E}}} \Omega_{\mathcal{E}} X)$ satisfies $\Omega_{\mathcal{E}} \overline{e_1} \otimes 1_{\Omega_{\mathcal{E}} X} \circ \Phi(1_{\Omega_{\mathcal{E}} X}) = \Omega_{\mathcal{E}} \overline{e_2} \otimes 1_{\Omega X} \circ \Phi(1_{\Omega_{\mathcal{E}} X})$ where $\overline{e_1}, \overline{e_2}, L$ are as in the standard presentation. Given $\alpha : \mathbb{S}^X \xrightarrow{\cdot} \mathbb{S}^W$, by naturality of such α it follows that

$$\Omega_{\mathcal{E}} \overline{e_1} \otimes 1_{\Omega_{\mathcal{E}} W} \circ \alpha_{Idl(L)}(\Phi(1_{\Omega_{\mathcal{E}} X})) = \Omega_{\mathcal{E}} \overline{e_2} \otimes 1_{\Omega_{\mathcal{E}} W} \circ \alpha_{Idl(L)}(\Phi(1_{\Omega_{\mathcal{E}} X}))$$

and so $\alpha_{Idl(L)}(\Phi(1_{\Omega_{\mathcal{E}}X}))$ factors through the equalizer $E^{W,X}$ and so corresponds to an $\overline{\alpha} : \Lambda^{\Omega_{\mathcal{E}}X} \dot{\to} \Lambda^{\Omega_{\mathcal{E}}W}$. Now by definition of γ_* and construction of $\overline{\alpha}$,

$$(\gamma_*\overline{\alpha})_{Idl(L)}(\Phi(1_{\Omega X})) = 1 \overset{\Phi(1_{\Omega X})}{\to} \mathcal{U}L \otimes_{\mathbf{Sup}_{\mathcal{E}}} \Omega X \overset{\overline{\alpha}_{\mathcal{U}L}}{\to} \mathcal{U}L \otimes_{\mathbf{Sup}_{\mathcal{E}}} \Omega W$$
$$= \alpha_{Idl(L)}(\Phi(1_{\Omega X}))$$

and so by Lemma 3.4, $\gamma_*\overline{\alpha} = \alpha$. The assignment $\alpha \longmapsto \overline{\alpha}$ is therefore an injection.

Given any $\overline{\alpha} : \Lambda^{\Omega_{\mathcal{E}}X} \dot{\to} \Lambda^{\Omega_{\mathcal{E}}W}$, certainly $\gamma_*\overline{\alpha} : \mathbb{S}^X \dot{\to} \mathbb{S}^W$. But $(\overline{\gamma_*\overline{\alpha}})_{\mathcal{U}L} \circ \Phi(1_{\Omega X}) = (\gamma_*\overline{\alpha})_{Idl(L)}(\Phi(1_{\Omega X})) = \overline{\alpha}_{\mathcal{U}L} \circ \Phi(1_{\Omega X})$. Therefore $\overline{(\gamma_*\overline{\alpha})} = \overline{\alpha}$ by Lemma 3.4 relative to $\mathcal{E}$ and so $\alpha \longmapsto \overline{\alpha}$ is a surjection.

Since $\gamma_*(\overline{\beta} \circ \overline{\alpha}) = \gamma_*(\overline{\beta}) \circ \gamma_*(\overline{\alpha})$ by an easy calculation, naturality is immediate.

(b) By construction $\gamma_* : \mathcal{E} \to \mathbf{Set}$ clearly preserves finite products and therefore as an action from $[\mathbf{Loc}_{\mathcal{E}}^{op}, \mathcal{E}]$ to $[\mathbf{Loc}_{\mathcal{E}}^{op}, \mathbf{Set}]$, γ_* preserves finite products given their componentwise construction in these functors categories. γ_* therefore takes internal distributive lattices to internal distributive lattices and so $\mathbb{S}^X = \gamma_*\Lambda^{\Omega_{\mathcal{E}}X}$ is an internal distributive lattice. For example the join natural transformation $\sqcup : \mathbb{S}^X \times \mathbb{S}^X \dot{\to} \mathbb{S}^X$ is $\gamma_*\sqcup^{\Lambda^{\Omega_{\mathcal{E}}X}}$ where $\sqcup^{\Lambda^{\Omega_{\mathcal{E}}X}} : \Lambda^{\Omega_{\mathcal{E}}X} \times \Lambda^{\Omega_{\mathcal{E}}X} \dot{\to} \Lambda^{\Omega_{\mathcal{E}}X}$ is the join on $\Lambda^{\Omega_{\mathcal{E}}X}$.

To prove (b) it is sufficient to verify that under the bijection given in (a), distributive lattice homomorphisms $\Lambda^{\Omega_{\mathcal{E}}X} \dot{\to} \Lambda^{\Omega_{\mathcal{E}}W}$ correspond to distributive lattice homomorphisms $\mathbb{S}^X \dot{\to} \mathbb{S}^W$. Clearly $\gamma_*\overline{\alpha}$ is a distributive lattice homomorphism if $\overline{\alpha}$ is since, as we have just indicated, γ_* preserves products.

In the other direction we need to recall that for any frames ΩX, ΩY and ΩZ,

$$\Omega Y \otimes_{\mathbf{Sup}} (\Omega X \times \Omega Z) \cong \Omega Y \otimes_{\mathbf{Sup}} \Omega X \times \Omega Y \otimes_{\mathbf{Sup}} \Omega Z.$$

Locally (i.e. topologically) this is just the assertion that the category of locales is distributive (i.e. $Y \times (X+Z) \cong Y \times X + Y \times Z$). It can be proved lattice theoretically by noting that $\Omega X \times \Omega Z$ is suplattice coproduct and using bijections such as $\mathbf{Sup}(\Omega Y \otimes_{\mathbf{Sup}} \Omega X, \Omega W) \cong \mathbf{Sup}(\Omega Y, [\Omega X, \Omega W])$ where $[\Omega X, \Omega W]$ is the suplattice of suplattice homomorphisms from ΩX to ΩW. This bijection is natural in ΩY and so (taking $\Omega Z = \Omega X$) the conclusion needed for

our proof is that

$$\Lambda^{\Omega_\mathcal{E} X} \times \Lambda^{\Omega_\mathcal{E} X} \cong \Lambda^{\Omega_\mathcal{E} X \times \Omega_\mathcal{E} X}.$$

Therefore given an $\overline{\alpha} : \Lambda^{\Omega_\mathcal{E} X} \dashrightarrow \Lambda^{\Omega_\mathcal{E} W}$ it follows that to prove, for example, that

$$
\begin{array}{ccc}
\Lambda^{\Omega_\mathcal{E} X} \times \Lambda^{\Omega_\mathcal{E} X} & \xrightarrow{\overline{\alpha} \times \overline{\alpha}} & \Lambda^{\Omega_\mathcal{E} W} \times \Lambda^{\Omega_\mathcal{E} W} \\
\sqcup^{\Lambda^{\Omega X}} \downarrow & & \downarrow \sqcup^{\Lambda^{\Omega W}} \\
\Lambda^{\Omega_\mathcal{E} X} & \xrightarrow{\overline{\alpha}} & \Lambda^{\Omega_\mathcal{E} W}
\end{array}
$$

commutes it is sufficient to prove that it commutes when evaluated at $\Phi(1_{\Omega_\mathcal{E} X \times \Omega_\mathcal{E} X})$ at $\mathcal{U}(L \times L)$ (where $L \times L = \Omega_\mathcal{E} X \times \Omega_\mathcal{E} X$, the generators of $\Omega_\mathcal{E} X \times \Omega_\mathcal{E} X$ in the standard presentation). But if the corresponding map $\gamma_* \overline{\alpha}$ is a join semilattice homomorphism then we know that the diagram commutes when evaluated at $x : 1 \to \Omega_\mathcal{E} Y \otimes_{\mathbf{Sup}_\mathcal{E}} (\Omega_\mathcal{E} X \times \Omega_\mathcal{E} X)$ for every x and every $\Omega_\mathcal{E} Y$.

It follows that if α is a distributive lattice homomorphism then so is $\overline{\alpha}$. $\qquad\qquad\square$

7. Summary

This paper is meant as a write up of original ideas contained in [TV03], here using frame theory to guide the proofs of results, and then reinterpreting to give the localic content.

Given that dcpo presentations present (folklore, though here shown constructively), one is able to view suplattices via dcpo presentations. In particular frame coproduct (i.e. locale product) which is known to be presented by suplattice tensor, can be described as a free dcpo (subject to relations). Given this any dcpo homomorphism between frames, $\Omega X \to \Omega W$, can be extended to a dcpo homomorphism $\Omega Y +_{\mathbf{Fr}} \Omega X \to \Omega Y +_{\mathbf{Fr}} \Omega W$ for any frame ΩY, and this extension is natural, in that it commutes with the frame homomorphisms on the ΩY. In categorical language this defines a natural transformation from the functor $(\_) +_{\mathbf{Fr}} \Omega X$ to $(\_) +_{\mathbf{Fr}} \Omega W$.

By using a dcpo presentation for the frame ΩX, dcpo homomorphisms can be described as monotone maps which satisfy certain relations. 'Satisfaction' means composing equally with two dcpo homomorphism between ideal completions, but by a well known correspondence (essentially the assertion from locale theory that algebraic dcpos are spatial) such dcpo homomorphisms are equivalently

frame homomorphism between frames of upper closed subsets. Using this it can be shown that a dcpo homomorphism $\Omega X \to \Omega W$ is exactly an element of the set $\mathcal{U}L +_{\mathbf{Fr}} \Omega W$ that composes equally with two frame homomorphism, where L is a lattice of generators for ΩX and $\mathcal{U}L$ is the set of upward closed subsets of L. But the identity map on ΩX provides a natural 'identity element' in $\mathcal{U}L +_{\mathbf{Fr}} \Omega X$ and so any natural transformation from $(\_) +_{\mathbf{Fr}} \Omega X$ to $(\_) +_{\mathbf{Fr}} \Omega W$ gives rise to an element of $\mathcal{U}L +_{\mathbf{Fr}} \Omega W$ by application at $\mathcal{U}L$. This element, by naturality, will compose equally with the two frame homomorphism and so gives rise to a dcpo homomorphism.

These constructions are in bijection and so the dcpo homomorphisms can be represented by natural transformations. In fact the functors $(\_) +_{\mathbf{Fr}} \Omega X$ are internal distributive lattices in the category of all functors from **Fr** to **Set** and this result specializes to show that frame homomorphism are exactly internal distributive lattice homomorphism. Since, via the localic account of topology, frame homomorphisms correspond to continuous maps, this result gives a representation theorem for a well known notion of continuity.

References

[J82]   Johnstone, P.T. *Stone Spaces*. Cambridge Studies in Advanced Mathematics **3**. Cambridge University Press, 1982.

[J02]   Johnstone, P.T. *Sketches of an elephant: A topos theory compendium*. Vols 1, 2, Oxford Logic Guides **43**, **44**, Oxford Science Publications, 2002.

[JV91]   Johnstone, P.T. and Vickers, S.J. "Preframe presentations present", in Carboni, Pedicchio and Rosolini (eds) Category Theory – Proceedings, Como, 1990 (*Springer Lecture Notes in Mathematics*, **1488**, 1991), 193-212.

[JT84]   Joyal, A. and Tierney, M. An Extension of the Galois Theory of Grothendieck, *Memoirs of the American Mathematical Society* **309**, 1984.

[MacL71]   Mac Lane, S. "Categories for the Working Mathematician". GTM **5**. Springer-Verlag. (1971).

[M77]   Markowsky, George "Categories of chain-complete posets". *Theoret. Comput. Sci.* **4** (1977), 125-135.

[T96]   Townsend, C.F. *Preframe Techniques in Constructive Locale Theory*, Phd Thesis, 1996, Imperial College, London.

[T04]   Townsend, C.F. *An Axiomatic account of Weak Localic Triquotient Assignments*. To appear. Journal of Pure and Applied Algebra.

CHRISTOPHER TOWNSEND

[TV03]   Townsend, C.F. and Vickers, S.J. *A Universal Characterization of the Double Power Locale.* Theoretical Computer Science 316 (2004) 297-321.

[V93]   Vickers, S.J. "Information systems for continuous posets". *Theoretical Computer Science* **114** (1993), 201–229.

8 AYLESBURY ROAD, TRING, HERTFORDSHIRE, ENGLAND, HP23 4DJ
E-mail address: cft71@hotmail.com