An Axiomatic account of Weak Localic Triquotient Assignments DRAFT

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Abstract

Localic triquotient maps were studied by Plewe as a natural generalization of open and proper localic surjections. Using a weaker notion of triquotient assignment, introduced by Vickers, arbitrary proper and open locale maps can be studied. This paper gives an axiomatic account of the theory of weak triquotient assignments, recovering what is known localically: triquotient surjections are of effective descent, proper and open maps are pullback stable. Triquotient inclusions are also characterized.

The axiomatic account covers compact Hausdorff and discrete objects, and it is shown that these form regular categories, a fact observed by Taylor using a different axiomatization.

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1 Introduction

This paper offers an axiomatic account of triquotient maps in locale theory recovering the main results known of these maps: they are pullback stable and triquotient surjections are of effective descent. It is also shown how to recover an axiomatic theory of open and proper maps using weak triquotient assignments, thereby giving an axiomatic account of discrete and compact Hausdorff objects. The key is a representation theorem for weak triquotient assignments in terms of natural transformations.

In topology a triquotient surjection, $p : Z \to Y$, is a map with a triquotient assignment; that is an assignment

$$p_{\#} : Opens(Z) \to Open(Y),$$

required to satisfy certain conditions. These maps were originally studied in point set topology by [Michael 77] as a useful class of pullback stable coequalizers. Localic triquotient assignments were introduced by Plewe ([Plewe 97]) in his localic description of the triquotient surjections from topology. Localic triquotient surjections conservatively extend topological triquotient surjections between spaces provided the codomain space is T_D (that is, provided each point is an open subspace of its closure). It is shown in [Plewe 97] that localic triquotient surjections are effective descent maps for the category of locales, a result which generalized the same conclusion for open surjections ([JoyTie 84]) and proper surjections ([Vermeulen 93]).

Subsequently a weaker notion of triquotient assignment was introduced by Vickers [Vickers 01a], and it is this notion that we shall explore in this paper. In contrast to Plewe's definition this notion does not force a triquotient map to be a surjection, indeed every locale map has a weak triquotient assignment. Vickers, [Vickers 01a], conjectured that the weak triquotient assignments on a locale map $p : Z \to Y$ are exactly the points of the double power locale on $p : Z \to Y$ relative to the topos of sheaves over Y. This was proved in [Townsend 03] and in [TowVic 02] it is shown that the points of the double power locale on $p : Z \to Y$ can be represented by maps in the presheaf category

$$[(\mathbf{Loc}/Y)^{op}, \mathbf{Set}]$$

i.e. they can be represent by certain natural transformations, where $\mathbf{Loc}/Y$ is the category of locales sliced over Y. Combining these observations we have that weak triquotient assignments can be represented by natural transformations, and the main technical result of this paper is an axiomatic account of this representation theorem. From this our key results follow by reasonably straightforward categorical arguments. The key results are

(a) weak triquotient assignments are pullback stable and satisfy Beck-Chevalley,

(b) triquotient surjections are pullback stable and of effective descent,

(c) triquotient inclusions are exactly the finite meets of open and closed subobjects; and

(d) the categories of discrete objects and of compact Hausdorff objects are both regular.

(a), (b) and (c) have not been shown axiomatically though all are known for the category of locales. An axiomatic account of (d) does exist in [Taylor 00], but our approach using weak triquotient assignments is new.

The analysis also covers some of the basic relationships familiar from locale theory and sheaf theory: (i) a locale map $p : X \to Y$ is proper if and only if it is compact as a locale internal to the category of sheaves over Y (denoted $Sh(Y)$) and (ii) for any topological space X, $\mathbf{LH}/X \simeq Sh(X)$ where $\mathbf{LH}/X$ is the category of local homeomorphisms over X.

The category of locales is our canonical example of a category which satisfies the axioms. The category of locales has been investigated by a number of authors ([Johnstone 82]) and can be considered as a good framework for constructive topology; that is, an account of topology without an assumption of the excluded middle. An advantage of the category of locales is the greater logical generality but a disadvantage is that not every result about topological spaces has a clear localic analogue. However we also show that the category of finite posets is a model for our axioms. Using this model, and borrowing techniques of Janalidze and Sobral ([JanSob 02]) we prove that the axioms do not imply that every effective descent morphism is a triquotient surjection; this is in keeping with the topological case.

An achievement of the axioms offered in this paper is that the observed parallel between proper and open maps (e.g. [Townsend 96], [Vickers 95] and [Vermeulen 93]) can now be expressed as a formal duality. One of the axioms will be that the category is order enriched and all the other axioms are stable under order duality. The axiomatic theory of open maps is identical to the axiomatic theory of proper maps in the order dual. It follows that the axiomatic theory compact Hausdorff locales is identical to the axiomatic theory of discrete locales ([Taylor 00] also enjoys this duality). Of course it would be difficult to argue, topologically, that the theory of compact Hausdorff spaces is *the same thing as* the theory of discrete spaces (i.e. set theory) and so this presents a barrier to further work giving an entirely axiomatic account of topology. However the axioms of this paper represent a general formal account of the proper/open parallel.

The final section gives an axiomtic proof of Vickers' result about localic locales, [Vickers 01b]. The study of localic locales is the study of locale like objects internal to the category of locales. More formally the category of localic locales is the opposite of the category of algebras of the double power locale monad. The final section begins with an axiomatic account of the double power locale which is achieved by strengthening one of the axioms.

2 Locales, Dcpos and Natural Transformations

Let us recall some definitions and notation from locale theory. We assume familiarity with basic lattice theoretic and categorical definitions and notation (see, for example, [Johnstone 82] and [MacLane 71]). A frame is a complete lattice which satisfies the distributivity law

$$a \wedge \bigvee T = \bigvee \{a \wedge t \mid t \in T\}$$

for any element a and subset T. A frame homomorphism preserves arbitrary joins and finite meets and so a category, **Fr**, of frames is defined. A frame will be denoted ΩX, where X is the *corresponding locale.* This comes from the definition of the category of locales:

$$\mathbf{Loc} \equiv \mathbf{Fr}^{op},$$

that is, the category of locales is taken to be the dual of the category of frames. There is therefore notational but no mathematical difference between a locale and a frame. Given a locale map $f : X \to Y$ (i.e. the localic notion of a continuous map between spaces) the corresponding frame homomorphism is denoted $\Omega f : \Omega Y \to \Omega X$. For every topological space, X, a locale can be defined by $\Omega X \equiv \mathit{Opens}(X)$. So, notationally, we also use X for the locale corresponding to the space. A continuous map between topological spaces therefore gives rise to a locale map.

The category of frames can also be described using the category, **dcpo**, of directed complete partial orders with directed join preserving maps as morphisms (i..e Scott continuous maps). It is cartesian closed and a routine calculation shows that dcpo tensor is given by binary product (i.e. $A \otimes B \equiv A \times B$ for any dcpos A, B). From this it can be shown that the category of all order internal distributive lattices, denoted **DLat**(**dcpo**), is isomorphic to the category of frames. The expression 'order-internal' in this context means that the meet(join) operation is required to be the right(left) adjoint to the diagonal in the order enrichment. In summary,

$$\mathbf{Loc} \cong \mathbf{DLat}(\mathbf{dcpo})^{op}.$$

Now coequalizers exist in the category of dcpos (widely known, see, for example, [TowVic 02]) and from this the locale equalizer of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ can be calculated: it is the locale map whose corresponding frame homomorphism is given by the dcpo coequalizer of

$$\Omega X \times \Omega X \times \Omega Y \overset{1\times 1\times \Omega f}{\underset{1\times 1\times \Omega g}{\rightrightarrows}} \Omega X \times \Omega X \times \Omega X \overset{1\times\vee}{\rightarrow} \Omega X \times \Omega X \overset{\wedge}{\rightarrow} \Omega X$$

where $\wedge, \vee : \Omega X \times \Omega X \rightarrow \Omega X$ are the binary meet, join operations. A categorical proof of this has been included as an Appendix, where it is shown how to calculate order enriched coequalizers in **DLat**(**C**) from order enriched coequalizers in **C** for any order enriched cartesian closed **C**. By 'order enriched' colimit or limit we mean that the bijection contained in the definition of colimit/limit must be order preserving. For the example of order enriched binary product this means that if X, Y are objects in an order enriched category **C** then for any third object W the bijection

$$\mathbf{C}(W, X) \times \mathbf{C}(W, Y) \overset{(\_,\_)}{\rightarrow} \mathbf{C}(W, X \times Y)$$

is required to be order preserving. Dcpo coequalizers are order enriched.

Locale products can be described using suplattice tensor, [JoyTie 84]. A suplattice is a complete lattice and a suplattice homomorphism is an arbitrary join preserving map. For any locales X, Y

$$\Omega(X \times Y) \equiv \Omega X \otimes_{\mathbf{Sup}} \Omega Y$$

where **Sup** is the category of suplattices. The diagonal map $\Delta : X \hookrightarrow X \times X$ is defined by $\Omega\Delta(a_1 \otimes a_2) = a_1 \wedge a_2$. The initial frame, Ω, is defined by $\Omega = P\{*\}$; that is, the power set of the singleton set. Classically $\Omega \cong \{0 \leq 1\}$, but this is not true without an assumption of the excluded middle.

Although the theory of frames contains an infinitary join operation it is suitably algebraic since coequalizers exist, and free frames can be constructed, see e.g. Ch. II in [Johnstone 82]. Frame limits (i.e. locale colimits) are created in **Set** and can be seen to be order enriched. Frame presentations present, for example the frame

$$\Omega\mathbb{S} \equiv \mathbf{Fr}\langle\{*\}\rangle$$

is well defined; that is, the free frame on a single generator. $\Omega\mathbb{S}$ can be described more concretely as the set of monotone maps from $\{0 \leq 1\} \to \Omega$. Classically, therefore, $\Omega\mathbb{S}$ is the three point chain. The corresponding locale $\mathbb{S}$ is known as the Sierpiński locale and will play a central role in what follows. The category of locales is order enriched (it inherits this from **dcpo**) and $\mathbb{S}$ is an order internal distributive lattice. This follows by a routine calculation from the definitions of $1_{\mathbb{S}} : 1 \to \mathbb{S}$, $\sqcap : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$, $0_{\mathbb{S}} : 1 \to \mathbb{S}$ and $\sqcup : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$ by

$$\begin{aligned} \Omega 1_{\mathbb{S}}(*) &= 1_{\Omega} \\ \Omega \sqcap (*) &= * \otimes * \\ \Omega 0_{\mathbb{S}}(*) &= 0_{\Omega} \\ \Omega \sqcup (*) &= * \otimes 1 \vee 1 \otimes *. \end{aligned}$$

The category of locales enjoys the following slice stability property proved in Joyal and Tierney's paper [JoyTie 84],

$$\mathbf{Loc}/Z \simeq \mathbf{Loc}_{Sh(Z)}$$

for any locale Z, where $\mathbf{Loc}_{Sh(Z)}$ is the category of locales internal to the topos of sheaves over Z. Therefore any constructive statement true of the category of locales is true of the slice $\mathbf{Loc}/Z$ for any locale Z. For example the Sierpiński locale exists relative to Z: it is given by $\pi_1 : Z \times \mathbb{S} \to Z$ for which we will use the notation $\mathbb{S}_Z$.

A result in [TowVic 02] shows that for any locales Y and X, dcpo homomorphisms $\Omega Y \to \Omega X$ are in natural order isomorphism with

$$Nat[\mathbf{Loc}(\_ \times Y, \mathbb{S}), \mathbf{Loc}(\_ \times X, \mathbb{S})]$$

where $\mathbf{Loc}(\_ \times Y, \mathbb{S}) : \mathbf{Loc}^{op} \to \mathbf{Set}$ is the presheaf for any locale Y and $Nat[\_]$ is the collection of natural transformations ordered componentwise in the obvious manner. This isomorphism is an extension, to dcpo homomorphisms, of the obvious mapping:

$$\Omega f \longmapsto \mathbf{Loc}(\_ \times f, \mathbb{S})$$

for any frame homomorphism $\Omega f : \Omega Y \to \Omega X$. The proof of this, it is shown in [TowVic 02], holds relative to any locally small topos, so combining with the Joyal and Tierney correspondence we have that there is an order isomorphism between

$$Nat[\mathbf{Loc}/Z(\_ \times Y, \mathbb{S}_Z), \mathbf{Loc}/Z(\_ \times X, \mathbb{S}_Z)]$$

and dcpo maps $\Omega_{Sh(Z)}Y \to \Omega_{Sh(Z)}X$ for any locales X, Y in $Sh(Z)$. Here $\mathbf{Loc}/Z(\_ \times Y, \mathbb{S}_Z) : (\mathbf{Loc}/Z)^{op} \to \mathbf{Set}$.

This representation of dcpo maps relative to any locale is the starting point for the ideas in this paper. Once we have a suitable axiomatization for $\mathbb{S}$, which will be slice stable, then we have a categorical account for dcpo maps between frames: they are certain natural transformations. The remainder of the axiomatization concerns itself with ensuring that the relationship between these natural transformations and continuous maps mimics (in every slice) the known relationship between dcpo maps and locale maps. The known relationships that we mimic axiomatically are

(a) dcpo isomorphisms between frames necessarily correspond to locale isomorphisms (which is easy lattice theory),

(b) locale coequalizers give rise to dcpo equalizers (which holds since both frame and dcpo equalizers are calculated in the background set theory), and

(c) locale equalizers can always be calculated via dcpo coequalizers (as described above).

3 The Axioms

We now present axioms on a category $\mathbf{C}$, and discuss how the category of locales satisfies each. The proofs are all in the literature.

Axiom 1 $\mathbf{C}$ *is an order enriched category with order enriched finite limits and order enriched finite colimits.*

This is well known for locales (e.g. [Johnstone 82], [Vic 89]), and we have outlined how limits and colimits are constructed in the previous section. It should be clear, from construction, that the limits and colimits for $\mathbf{C} = \mathbf{Loc}$ are all order enriched. For any objects Z of $\mathbf{C}$, the forgetful functor $\mathbf{C}/Z \to \mathbf{C}$ creates finite order enriched limits and colimits, and so this axiom is stable under slicing.

Axiom 2 *For any objects* X, Y *and* W *in* $\mathbf{C}/Z$, $X \times (Y + W) \cong X \times Y + X \times W$. *Further* $X \times 0 \cong 0$.

This is a known distributivity law for the category of locales. To prove it, recall that $\Omega(Y + X) \equiv \Omega Y \times \Omega X$, i.e. locale coproduct is given by taking the set-theoretic product of the underlying frames. But $\Omega Y \times \Omega X$ can be verified to be suplattice coproduct of ΩY and ΩX. This distributivity law for locales then follows by a routine universal argument using the suplattice tensor description of locale product and the fact that the category of suplattices is cartesian closed. The axiom is true in $\mathbf{Loc}/Z$ for any Z since the argument is constructive and so valid in $Sh(Z)$.

Axiom 3 *There is a non-trivial order-internal distributive lattice denoted* $\mathbb{S}$ *such that given a pullback*

$$\begin{array}{ccc} a^*(i) & \rightarrow & 1 \\ \downarrow & & \downarrow i \\ X & \xrightarrow{a} & \mathbb{S} \end{array}$$

a is uniquely determined by $a^*(i) \hookrightarrow X$ for $i : 1 \to \mathbb{S}$ equal to either $0_\mathbb{S}$ or $1_\mathbb{S}$.

This is observed in Ch. 8 [Townsend 96] for the category of locales. That $\mathbb{S}$ is an order internal distributive lattice is observed above. By the definition of $\mathbb{S}$ via the free frame on a single generator we have that $\mathbf{Loc}(X, \mathbb{S}) \cong \Omega X$. For $i = 1_\mathbb{S} : 1 \to \mathbb{S}$ therefore note that a square

$$\begin{array}{ccc} W & \to & 1 \\ f\downarrow & & \downarrow 1_\mathbb{S} \\ X & \overset{a}{\to} & \mathbb{S} \end{array}$$

commutes in **Loc** if and only if $\Omega f(a) = 1_W$, and that this is true if and only if $\Omega f : \Omega X \to \Omega W$ factors via the frame homomorphism $a \wedge (\_) : \Omega X \to \downarrow a$. Therefore $\downarrow a$ is the frame corresponding to the pullback of $1_\mathbb{S} : 1 \to \mathbb{S}$ along a, i.e. $\Omega a^*(1_\mathbb{S}) \cong \downarrow a$. But $\downarrow a_1 = \downarrow a_2$ (in $Sub(X)$) implies $a_1 = a_2$, and so the axiom is proved for locales when $i = 1_\mathbb{S} : 1 \to \mathbb{S}$. For $i = 0_\mathbb{S} : 1 \to \mathbb{S}$ use an identical argument with $a \vee (\_) : \Omega X \to \uparrow a$ in the place of $a \wedge (\_) : \Omega X \to \downarrow a$.

We refer to $\mathbb{S}$ as a Sierpiński object if it satisfies this axiom. Without non-triviality the terminal object 1 would always be a Sierpiński object. A variation on this axiom also, in effect, appears in [Taylor 00] via Definition 2.2. The category of locally compact locales satisfies Axiom 3 as it has all pullbacks of the form required. However it does not have arbitrary equalizers and so is not a model for our axioms. Note further that since a Sierpiński object is required to be an order-internal distributive lattice it is not the case, even with an assumption of the excluded middle, that the subobject classifier Ω is a Sierpiński object relative to **Set**. Aside from $\{0 \leq 1\}$ in the category of posets, **Pos**, we have been unable to exhibit a non-standard Sierpiński object even with an assumption of the excluded middle. Thus it has not been possible to prove each of the following axioms independent. This paper is therefore only offering a possible axiomatization of **Loc**; sharper axiomatizations may exist.

The property of having a Sierpiński object is stable under slicing. It can be verified that $\mathbb{S}_Z \equiv \pi_1 : Z \times \mathbb{S} \to Z$ is a Sierpiński object in $\mathbf{C}/Z$ if $\mathbb{S}$ is a Sierpiński object in $\mathbf{C}$. To see this note that

$$\begin{array}{ccc} Z \times 1 & \overset{!}{\to} & 1 \\ 1 \times 1_\mathbb{S}\downarrow & & 1_\mathbb{S}\downarrow \\ Z \times \mathbb{S} & \overset{\pi_2}{\to} & \mathbb{S} \end{array}$$

is a pullback and to prove that $a, b : W \to \mathbb{S}_Z$ are equal in $\mathbf{C}/Z$ it is sufficient to show $\pi_2 a = \pi_2 b$.

In the next axiom, for any object Z, and for any object X in $\mathbf{C}/Z$ we use the notation $\mathbb{S}_Z^X$ for the functor

$$\begin{array}{rcl} (\mathbf{C}/Z)^{op} & \to & \mathbf{Set} \\ Y & \longmapsto & \mathbf{C}/Z(Y \times X, \mathbb{S}_Z) \end{array}$$

It can be verified, using Yoneda's lemma, that $\mathbb{S}_Z^X$ is the exponential

$$\mathbf{C}/Z(\_, \mathbb{S}_Z)^{\mathbf{C}/Z(\_, X)}$$

in the presheaf category $[(\mathbf{C}/Z)^{op}, \mathbf{Set}]$. All the presheaves we consider are either representable or of the form $\mathbb{S}_Z^X$; the notation $\mathbf{C}_Z^{op}$ denotes the full subcategory of $[(\mathbf{C}/Z)^{op}, \mathbf{Set}]$ consisting of objects of the form $\mathbb{S}_Z^X$. By an application of Axiom 2 we have that $\mathbf{C}_Z^{op}$ is closed under binary products since

$$\begin{aligned} \mathbb{S}_Z^{X+Y} &\cong \mathbb{S}_Z^X \times \mathbb{S}_Z^Y \text{ and} \\ \mathbb{S}_Z^0 &\cong 1, \end{aligned}$$

for any X, Y in $\mathbf{C}/Z$. Note that therefore $\mathbb{S}_Z^X$ is an order internal distributive lattice in $\mathbf{C}_Z^{op}$ for any X in $\mathbf{C}/Z$; the distributive lattice structure comes from $\mathbb{S}$ and the ordering on natural transformations is the obvious pointwise ordering. The order enrichment on $\mathbf{C}_Z^{op}$ will be repeatedly exploited in what follows.

For the case $\mathbf{C} = \mathbf{Loc}$, given the relationship between dcpo homomorphisms and natural transformations discussed in the previous section we have that $\mathbf{Loc}_Z^{op}$ is equivalent to the category of frames relative to sheaves over Z, with morphisms all dcpo homomorphisms. This is because the mapping

$$\Omega_{Sh(Z)} X \longmapsto \mathbb{S}_Z^X$$

is the object part of a full, faithful and essentially surjective functor to $\mathbf{Loc}_Z^{op}$. The functor $\mathbb{S}_Z^{(\_)} : \mathbf{Loc}/Z \to \mathbf{Loc}_Z^{op}$ is then equivalent to one which sends a locale map, f, relative to Z, to the dcpo homomorphism $\Omega_{Sh(Z)} f$.

Axiom 4 *For any object Z, the contravariant functor $\mathbb{S}_Z^{(\_)} : \mathbf{C}/Z \to \mathbf{C}_Z^{op}$ (a) takes coequalizers to equalizers and (b) reflects isomorphisms.*

This axiom, for locales over Z, is the statement that (a) frame equalizers are calculated as dcpo equalizers and (b) if $\Omega_{Sh(Z)} Y \cong_{\mathbf{dcpo}} \Omega_{Sh(Z)} X$ via $\Omega_{Sh(Z)} f$, then $X \cong Y$ via f. Since a dcpo isomorphism is a frame isomorphism (b) is clear. Part (a) is immediate for locales since both frame and dcpo equalizers are created in the background set theory and so frame equalizers are dcpo equalizers. (Here, this set theory is $Sh(Z)$.)

The final axiom reflects the relationship between locale equalizers and dcpo coequalizers as outlined in the previous section and proved in the Appendix. Locale equalizers are calculated via a dcpo coequalizer construction. This relationship is introduced as the 'double coverage theorem' in [TowVic 02] as it is an extension of Johnstone's coverage result (II 2.11 in [Johnstone 82]), though note Section 5.2 in [AbrVic 93] for the key observation of a relationship between different classes of coequalizers (there between frame coequalizers and suplattice coequalizers).

Axiom 5 *For any equalizer diagram*

$$E \overset{e}{\hookrightarrow} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

in $\mathbf{C}/Z$ *the diagram*

$$\mathbb{S}_Z^X \times \mathbb{S}_Z^X \times \mathbb{S}_Z^Y \overset{\sqcap(1\times\sqcup)(1\times 1\times \mathbb{S}_Z^f)}{\underset{\sqcap(1\times\sqcup)(1\times 1\times \mathbb{S}_Z^g)}{\rightrightarrows}} \mathbb{S}_Z^X \xrightarrow{\quad \mathbb{S}^e \quad} \mathbb{S}_Z^E$$

is a coequalizer in $\mathbf{C}_Z^{op}$ *where* $\sqcap(1\times\sqcup)$ *is the composite*

$$\mathbb{S}_Z^X \times \mathbb{S}_Z^X \times \mathbb{S}_Z^X \xrightarrow{1\times\sqcup} \mathbb{S}_Z^X \times \mathbb{S}_Z^X \xrightarrow{\sqcap} \mathbb{S}_Z^X.$$

This is the final axiom, and so it is clear that they are all slice stable by construction. Note this final axiom implies that $\mathbb{S}_Z^{(\_)} : \mathbf{C}/Z \to \mathbf{C}_Z^{op}$ takes coreflexive equalizers to coequalizers since if $q : Y \to X$ splits f and g then $\sqcap(1\times\sqcup)(1\times 1\times \mathbb{S}_Z^f)$ and $\sqcap(1\times\sqcup)(1\times 1\times \mathbb{S}_Z^g)$ factor as

$$\mathbb{S}_Z^f \sqcap (1\times\sqcup)(\mathbb{S}_Z^q \times \mathbb{S}_Z^q \times Id) \text{ and}$$
$$\mathbb{S}_Z^g \sqcap (1\times\sqcup)(\mathbb{S}_Z^q \times \mathbb{S}_Z^q \times Id)$$

respectively. However it does not appear possible to conclude that this axiom is equivalent to the assertion "$\mathbb{S}_Z^{(\_)} : \mathbf{C}/Z \to \mathbf{C}_Z^{op}$ takes coreflexive equalizers to coequalizers". (Although every equalizer is isomorphic to a coreflexive equalizer, we do not know, for example, whether the obvious map $\mathbb{S}_Z^X \times \mathbb{S}_Z^X \times \mathbb{S}_Z^Y \to \mathbb{S}_Z^{Y\times X}$ is an epimorphism in $\mathbf{C}_Z^{op}$).

Given Axioms 4 and 5, $\mathbb{S}_Z^{(\_)} : \mathbf{C}/Z \to \mathbf{C}_Z^{op}$ is also faithful since if $g_1, g_2 : X \to Y$ in $\mathbf{C}/Z$ enjoy $\mathbb{S}_Z^{g_1} = \mathbb{S}_Z^{g_2}$ then $\mathbb{S}_Z^e$ is an isomorphism in $\mathbf{C}_Z^{op}$ by Axiom 5 where e is the equalizer of g_1 and g_2. Since by Axiom 4 $\mathbb{S}_Z^{(\_)}$ reflects isomorphisms, this is sufficient to show that e is an isomorphism, i.e. $g_1 = g_2$. Therefore $\mathbb{S}_Z^{(\_)} : \mathbf{C}/Z \to \mathbf{C}_Z$ is conservative (i.e. reflects isomorphisms and is faithful) and preserves coreflexive equalizers. If it had a right adjoint we could conclude that, via the dual of the Crude Monadicity Theorem, it is comonadic. For $\mathbf{C} = \mathbf{Loc}$ such a right adjoint exists since (see [TowVic 02]) the double exponential $\mathbb{S}_Z^{\mathbb{S}_Z^X}$ exists in $[(\mathbf{Loc}/Z)^{op}, \mathbf{Set}]$ as a representable functor and so $\mathbf{Loc}/Z$ is equivalent to a category of coalgerbas over $\mathbf{Loc}_Z^{op}$. Another result of this type is studied in the Double Power Monad section below.

The category of posets does not satisfy this axiom at $Z = 1$. For $\mathbf{C} = \mathbf{Pos}$ we can take $\mathbb{S} = \{0 \leq 1\}$. The preseheaf $\mathbb{S}^X$ is representable as the poset $\mathcal{U}X$, the set of upper closed subsets of the poset X. $\mathbf{Pos}_1^{op}$ is therefore the category whose objects are posets of the form $\mathcal{U}X$ and whose morphisms are monotone maps. Now

$$\phi \hookrightarrow \mathbb{N} \overset{(\_)+1}{\underset{Id}{\rightrightarrows}} \mathbb{N}$$

is an equalizer in $\mathbf{Pos}$ where ϕ is the empty poset and $\mathbb{N}$ is the natural numbers with its usual ordering. Now consider the map $k : \mathcal{U}\mathbb{N} \to \mathcal{U}1 \cong \Omega$, which sends a subset to 1 if and only if it is non-empty. Note that k is a distributive lattice homomorphism since if U_1 and U_2 are two non-empty upper closed subsets of $\mathbb{N}$ then their intersection is non-empty. But from this (and the fact that $\mathcal{U}((\_)+1)$ preserves non-emptiness and emptiness) we have that k composes equally with

$$\mathcal{U}\mathbb{N} \times \mathcal{U}\mathbb{N} \times \mathcal{U}\mathbb{N} \underset{\cap(1\times\cup)(1\times 1\times\mathcal{U}(Id))}{\overset{\cap(1\times\cup)(1\times 1\times\mathcal{U}((\_)+1))}{\rightrightarrows}} \mathcal{U}\mathbb{N}$$

from which k must factor through $! : \mathcal{U}\mathbb{N} \to \mathcal{U}\phi \cong 1$, if the axiom were true of **Pos**; this is not possible.

Finally note that Axiom 5 does not break the symmetry given by the order enrichment. A short calculation using the distributivity assumption on $\mathbb{S}$ shows that the composite $\sqcup(1\times\sqcap)$ could have been used in the place of $\sqcap(1\times\sqcup)$. The other axioms are obviously order dual and so we have:

Theorem 6 *If an ordered enriched category* $\mathbf{C}$ *satisfies the axioms then so does its order dual,* $\mathbf{C}^{co}$.

4 Finite Poset Model

In this section we outline a proof that in a Boolean topos (i.e. assuming the excluded middle) the category of finite posets, **FinPos**, is a model for the axioms.

By finite Stone duality (using the excluded middle) **FinPos** is equivalent to **FinLoc** the category of finite locales (i.e. locales X such that ΩX is finite). For a proof, see for example, Th. 9.4.2 in [Vic 89]. Now it can be shown (though we omit the details) that Joyal and Tierney's correspondence specializes to

$$\mathbf{FinLoc}/M \simeq \mathbf{FinLoc}_{Sh(M)}$$

for any finite locale M. By finite, relative to $Sh(M)$, we mean Kuratowski finite, see for example D5.4 [Johnstone 02]. Verifying the axioms for the category of finite locales is routine given that they have been verified for arbitrary locales and any relevant construction preserves finiteness in $Sh(M)$. Note that $\Omega_{Sh(M)}$ is finite by the (finite) Joyal and Tierney correspondence and so the power set construction, $P_{Sh(M)}$, internal to $Sh(M)$, preserves finiteness, see [JohLin 78]. From this it can be verified that the relevant lattice theoretic constructions (i.e. frame colimits, limits, $\Omega_{Sh(M)}\mathbb{S}_M$ etc) preserve finiteness. Therefore Axioms 1,2 and 3 are verified for **FinPos**.

It is the case that the representation theorem of dcpo homomorphisms between frames in terms of natural transformations specializes to finite locales in $Sh(M)$ for finite M. However we do not need the details of this specialization of the representation theorem to prove that **FinPos** satisfies Axioms 4 and 5. This is because every finite locale X, in $Sh(M)$, is locally compact and so, see VII.4 in [Johnstone 82], exponentiable in the category $\mathbf{Loc}_{Sh(M)}$. Therefore $\mathbb{S}_M^X$ in $[\mathbf{FinLoc}^{op}_{Sh(M)}, \mathbf{Set}]$ is naturally isomorphic to a representable functor and so by Yoneda's lemma,

$$\begin{aligned} & Nat_{[\mathbf{FinLoc}^{op}_{Sh(M)},\mathbf{Set}]}[\mathbb{S}^Y_M,\mathbb{S}^X_M] \\ \cong\ & \mathbf{FinLoc}_{Sh(M)}(\mathbb{S}^Y_M,\mathbb{S}^X_M) \\ \cong\ & \mathbf{Loc}_{Sh(M)}(\mathbb{S}^Y_M,\mathbb{S}^X_M) \\ \cong\ & Nat_{[\mathbf{Loc}^{op}_{Sh(M)},\mathbf{Set}]}[\mathbb{S}^Y_M,\mathbb{S}^X_M] \\ \cong\ & \mathbf{dcpo}_{Sh(M)}(\Omega_{Sh(M)}Y,\Omega_{Sh(M)}X). \end{aligned}$$

where the last line is by the representation theorem over all locales. Axioms 4 and 5 therefore hold for **FinPos** since they hold for **Loc**.

5 Change of Base

Let us recall how change of base works for the category of locales before we state and prove an axiomatic change of base result. The first aim is to outline how locale pullback can be described in terms of dcpo constructions. We only outline how this works, the details are in [Townsend 03].

If $f : X \to Z$ is a locale map then, by common abuse of notation,

$$f : Sh(X) \to Sh(Z)$$

is a geometric morphism from the topos of sheaves over X to the topos of sheaves over Y. The direct image of f can be shown to preserve the property of being an internal dcpo and further defines a functor

$$f_* : \mathbf{dcpo}_{Sh(X)} \to \mathbf{dcpo}_{Sh(Z)}.$$

Now [Townsend 03] shows that this functor has a left adjoint, $f^\#$, and so since every frame is a dcpo we have that for any frame $\Omega_{Sh(X)}(A)$ internal to $Sh(X)$ and any frame $\Omega_{Sh(Z)}(B)$ internal to $Sh(Z)$,

$$\mathbf{dcpo}_{Sh(X)}(f^\#\Omega_{Sh(Z)}(B),\Omega_{Sh(X)}(A)) \cong \mathbf{dcpo}_{Sh(Z)}(\Omega_{Sh(Z)}(B), f_*\Omega_{Sh(X)}(A)). \qquad (*)$$

Further it can be verified (i) that both f_* and $f^\#$ preserve the property of being a frame and of being a frame homomorphism and (ii) that the isomorphism $(*)$ preserves the property of being a frame homomorphism. Given these observations we have an adjunction

$$\mathbf{Fr}_{Sh(X)} \underset{f_*}{\overset{f^\#}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Fr}_{Sh(Z)}.$$

Now [JoyTie 84] shows that $\mathbf{Fr}_{Sh(X)} \simeq (\mathbf{Loc}/X)^{op}$ and from the details of the proof of this fact (e.g. C1.6 [Johnstone 02]) it can be shown that

$$\Sigma_f : \mathbf{Loc}/X \to \mathbf{Loc}/Z,$$

is equivalent to f_*^{op}, where Σ_f is the 'compose with f functor' i.e. $g \longmapsto f \circ g$. Hence, since pullback is by definition right adjoint to Σ_f, we have that $(f^{\#})^{op}$ defines pullback. It follows that locale pullback can be described as change of base using only the dcpo data involved. This extra generality has proved key in various new results for locales [Townsend 03], and we need this extra level of generality when discussing change of base axiomatically. Given that $f^{\#}$ is equivalent to pullback, taking $A \in \mathbf{Loc}/X$ and $B \in \mathbf{Loc}/Z$ and re-interpreting dcpo homomorphisms between frames as natural transformations, $(*)$ reads

$$Nat_{[(\mathbf{Loc}/X)^{op},\mathbf{Set}]}[\mathbb{S}_X^{f^*B}, \mathbb{S}_X^A] \cong Nat_{[(\mathbf{Loc}/Z)^{op},\mathbf{Set}]}[\mathbb{S}_Z^B, \mathbb{S}_Z^{\Sigma_f A}].$$

In other words, for the category of locales, for any change of base map $f : X \to Z$, the adjunction $\Sigma_f \dashv f^*$ embeds into an adjunction

$$\mathbf{Loc}_X^{op} \begin{array}{c} \xleftarrow{f^{\#}} \\ \perp \\ \xrightarrow[f_*]{} \end{array} \mathbf{Loc}_Z^{op}$$

via $\mathbb{S}_Z^{(\_)} : \mathbf{Loc}/Z \to \mathbf{Loc}_Z^{op}$. Our next lemma proves this axiomatically. Notice that no use will be made of the classifying aspects of the Sierpiński object; any object could take its place. Indeed this lemma is simple 'abstract nonsense' and as such is probably known.

Lemma 7 *If $f : X \to Z$ is a map in a category $\mathbf{C}$ with pullbacks and a given object $\mathbb{S}$ then the pullback adjunction $\Sigma_f \dashv f^*$ extends to an adjunction*

$$\mathbf{C}_X^{op} \begin{array}{c} \xleftarrow{f^{\#}} \\ \perp \\ \xrightarrow[f_*]{} \end{array} \mathbf{C}_Z^{op}$$

via $\mathbb{S}_Z^{(\_)} : \mathbf{C}/Z \to \mathbf{C}_Z^{op}$.

Proof. Take $f^{\#}(\mathbb{S}_Z^B)$ to be the functor composition

$$(\mathbf{C}/X)^{op} \xrightarrow{\Sigma_f} (\mathbf{C}/Z)^{op} \xrightarrow{\mathbb{S}_Z^B} \mathbf{Set}$$

which, via a routine calculation, is naturally isomorphic to

$$(\mathbf{C}/X)^{op} \xrightarrow{\mathbb{S}_X^{f^*B}} \mathbf{Set}.$$

Take $f_*(\mathbb{S}_X^A)$ to be the functor composition

$$(\mathbf{C}/Z)^{op} \xrightarrow{f^*} (\mathbf{C}/X)^{op} \xrightarrow{\mathbb{S}_X^A} \mathbf{Set}$$

which, via a routine calculation, is naturally isomorphic to

$$(\mathbf{C}/Z)^{op} \stackrel{\mathbb{S}_Z^{\Sigma_f A}}{\rightarrow} \mathbf{Set}.$$

It is clear from the definition that the two squares

$$\begin{array}{ccc} \mathbf{C}/X & \rightarrow & \mathbf{C}_X^{op} \\ \Sigma_f \downarrow \ \uparrow f^* & & f_* \downarrow \ \uparrow f^\# \\ \mathbf{C}/Z & \rightarrow & \mathbf{C}_Z^{op} \end{array}$$

commute. So, for the claim of adjunction $f^\# \dashv f_*$, let $\eta : 1 \dot{\rightarrow} f^*\Sigma_f$, $\epsilon : \Sigma_f f^* \dot{\rightarrow} 1$ be the unit and counit of the adjunction $\Sigma_f \dashv f^*$. Then define $\overline{\eta} : Id \dot{\rightarrow} f_* f^\#$, $\overline{\epsilon} : f^\# f_* \dot{\rightarrow} Id$, by

$$\overline{\eta}_{\mathbb{S}_Z^B} = \mathbb{S}_Z^B \stackrel{\mathbb{S}_Z^{\epsilon_B}}{\rightarrow} \mathbb{S}_Z^{\Sigma_f f^* B}$$

(for $B \in \mathbf{C}/Z$) and

$$\overline{\epsilon}_{\mathbb{S}_X^A} = \mathbb{S}_X^{f^*\Sigma_f A} \stackrel{\mathbb{S}_X^{\eta_A}}{\rightarrow} \mathbb{S}_X^A,$$

for $A \in \mathbf{C}/X$. The triangular identities for $\overline{\eta}$ and $\overline{\epsilon}$ are therefore immediate from the fact that they hold for ϵ and η. ■

As observed above, on objects $\mathbf{DLat}(\mathbf{C}_X^{op}) = \mathbf{C}_X^{op}$ since the order internal distributive lattice structure is inherited from the Sierpiński object. This structure is preserved by change of base in both directions as the next lemma shows. To prove this we will need the distributivity axiom on $\mathbf{C}$ (Axiom 2), and henceforth, for ease of exposition, $\mathbf{C}$ will always be a category satisfying all the axioms.

Lemma 8 *Given $f : X \rightarrow Z$, $B \in \mathbf{C}/Z$ then for $A \in \mathbf{C}/X$*

$$\begin{aligned} f_* \sqcap_{\mathbb{S}_X^A} &= \sqcap_{f_*\mathbb{S}_X^A},\ f_* 1_{\mathbb{S}_X^A} = 1_{f_*\mathbb{S}_X^A}, \\ f_* \sqcup_{\mathbb{S}_X^A} &= \sqcup_{f_*\mathbb{S}_X^A},\ f_* 0_{\mathbb{S}_X^A} = 0_{f_*\mathbb{S}_X^A}, \end{aligned}$$

and for $B \in \mathbf{C}/Z$

$$\begin{aligned} f^\# \sqcap_{\mathbb{S}_Z^B} &= \sqcap_{f^\#\mathbb{S}_Z^B},\ f^\# 1_{\mathbb{S}_Z^B} = 1_{f^\#\mathbb{S}_Z^B}, \\ f^\# \sqcup_{\mathbb{S}_Z^B} &= \sqcup_{f^\#\mathbb{S}_Z^B},\ f^\# 0_{\mathbb{S}_Z^B} = 0_{f^\#\mathbb{S}_Z^B}. \end{aligned}$$

Further if $f : X \rightarrow Y$ is a morphism in $\mathbf{C}$ then $!^Y_*(\overline{\eta}_{\mathbb{S}_Y^C}) = \mathbb{S}^{p^*f} : \mathbb{S}^Z \rightarrow \mathbb{S}^{X \times_Y Z}$ where $\overline{\eta}$ is the unit of $f^\# \dashv f_*$, $C = p : Z \rightarrow Y$ is an arbitrary object of the slice $\mathbf{C}/Y$ and $!^Y : Y \rightarrow 1$ is the unique map to the terminal object.

Proof. The assertions about f_* are immediate since f_* is a right adjoint and so preserves binary products. Recall that the meet and join operations are order internal and so are right and left adjoint to finite diagonals: if the finite diagonals are preserved so are their right/left adjoints provided it is the case, as we have here from construction, that the map f_* preserves order.

For the assertions about $f^\#$ it is sufficient to verify that $f^\#$ preserves binary products. This follows from Axiom 2 since

$$\begin{aligned} f^\#(\mathbb{S}_Z^B \times \mathbb{S}_Z^B) &\cong f^\#\mathbb{S}_Z^{B+B} \cong \mathbb{S}_Z^{f^*(B+B)} \\ &\cong \mathbb{S}_Z^{f^*B+f^*B} \cong f^\#(\mathbb{S}_Z^B) \times f^\#(\mathbb{S}_Z^B). \end{aligned}$$

and similarly when $B = 0$.

For the further part, note that the unit of the adjunction $f^\# \dashv f_*$ is given by the counit of adjunction $\Sigma_f \dashv f^*$. This counit, at $p : Z \to Y$, is the top arrow of

$$\begin{array}{ccc} X \times_Y Z & \stackrel{p^*f}{\to} & Z \\ f^*p \downarrow & & \downarrow p \\ X & \stackrel{f}{\to} & Y \end{array}$$

■

6 The Sierpiński Object

We make some observations that are consequences of the definition of $\mathbb{S}$ via Axiom 3. These types of results were initially observed by Taylor in [Taylor 00]. However here the conclusions are about natural transformations, whereas Taylor's are about continuous maps and (conceptually at least) cover only sober locally compact spaces.

Lemma 9 *If $a, b : X \to \mathbb{S}$ are two maps then*
(a) $\sqcap_{\mathbb{S}}(a,b)$ (open) classifies $a^*(1) \wedge_{Sub(X)} b^*(1)$ and
(b) $\sqcup_{\mathbb{S}}(a,b)$ (closed) classifies $a^*(0) \wedge_{Sub(X)} b^*(0)$.
The following are equivalent
(i) $a \sqsubseteq b$,
(ii) $a^*(1) \leq_{Sub(X)} b^*(1)$ and
(iii) $b^*(0) \leq_{Sub(X)} a^*(0)$.

From Axiom 3 any map with codomain $\mathbb{S}$ closed classifies a *closed subobject* and open classifies an *open subobject* via pullback of $1, 0$ respectively. However, it seems overburdening to distinguish the different types of classification and so the term “classifies” is used to cover either. Context should make it clear which is meant.

Proof. (a)

$$\begin{array}{ccc} 1 & \to & 1 \\ (1,1) \downarrow & & \downarrow 1 \\ \mathbb{S} \times \mathbb{S} & \stackrel{\sqcap}{\to} & \mathbb{S} \end{array}$$

is a pullback since for any $p_1, p_2 \in \mathbf{C}(Z, \mathbb{S})$, $p_1 \sqcap_{\mathbf{C}(Z,\mathbb{S})} p_2 = 1_{\mathbf{C}(Z,\mathbb{S})} \Longrightarrow p_1 = 1_{\mathbf{C}(Z,\mathbb{S})}$ and $p_2 = 1_{\mathbf{C}(Z,\mathbb{S})}$. It is routine to verify that the pullback of $(1,1) : 1 \to \mathbb{S} \times \mathbb{S}$ along $(a,b) : X \to \mathbb{S} \times \mathbb{S}$ is $a^*(1) \wedge_{Sub(X)} b^*(1)$.

(b) Similarly since

$$\begin{array}{ccc} 1 & \to & 1 \\ (0,0) \downarrow & & \downarrow 0 \\ \mathbb{S} \times \mathbb{S} & \stackrel{\sqcup}{\to} & \mathbb{S} \end{array}$$

is a pullback.

(i)$\Longleftrightarrow$(ii). $a^*(1) \leq_{Sub(X)} b^*(1)$ iff the subobject $a^*(1) \stackrel{i_{a^*(1)}}{\hookrightarrow} X$ factors via $b^*(1) \stackrel{i_{b^*(1)}}{\hookrightarrow} X$ which, by the definition of $b^*(1)$ as pullback, is the case iff $bi_{a^*(1)} = 1!^{a^*(1)}$. But $1!^{a^*(1)}$ is the top element of the external meet semilattice $\mathbf{C}(a^*(1), \mathbb{S})$ and so $a^*(1) \leq_{Sub(X)} b^*(1)$ iff $1!^{a^*(1)} \sqsubseteq bi_{a^*(1)}$.

Say $a \sqsubseteq b$, then since $1!^{a^*(1)} = ai_{a^*(1)}$, it follows that $1!^{a^*(1)} \sqsubseteq bi_{a^*(1)}$, and so $a^*(1) \leq_{Sub(X)} b^*(1)$.

Conversely say $a^*(1) \leq_{Sub(X)} b^*(1)$, then $bi_{a^*(1)} = 1!^{a^*(1)}$, and so the open subspace classified by $bi_{a^*(1)}$ (which is the pullback of $b^*(1)$ along $i_{a^*(1)}$, i.e. the meet of $a^*(1)$ and $b^*(1)$ in $Sub(X)$) is isomorphic to the subobject classified by $1!^{a^*(1)} = ai_{a^*(1)}$, i.e. $a^*(1) \stackrel{Id}{\to} a^*(1)$. In other words, $a^*(1) = a^*(1) \wedge b^*(1)$ in $Sub(X)$. But, $a^*(1) \wedge b^*(1)$ is classified by $\sqcap(a,b)$ and so $a = \sqcap(a,b)$, i.e. $a \sqsubseteq b$.

(i)$\Longleftrightarrow$(iii). (Symmetric argument.) $b^*(0) \leq_{Sub(X)} a^*(0)$ iff the subobject $b^*(0) \stackrel{i_{b^*(0)}}{\hookrightarrow} X$ factors via $a^*(0) \stackrel{i_{a^*(0)}}{\hookrightarrow} X$, iff $ai_{b^*(0)} = 0!^{b^*(0)}$ iff $ai_{b^*(0)} \sqsubseteq 0!^{b^*(0)}$. But $bi_{b^*(1)} = 0!^{b^*(0)}$ and so $a \sqsubseteq b$ implies $b^*(0) \leq_{Sub(X)} a^*(0)$. Conversely if $ai_{b^*(0)} = 0!^{b^*(0)}$ then $b^*(0) = a^*(0) \wedge b^*(0)$ in $Sub(X)$ and so $a \sqsubseteq b$ since $a^*(0) \wedge b^*(0)$ is classified by $\sqcup(a,b)$. ■

Lemma 10 *If $a, b : X \to \mathbb{S}$ are two maps then the following are equivalent*

(i) $a \sqsubseteq b$,

(ii) for all $x_Z : Z \to X$ if ax_Z factors through $1 : 1 \to \mathbb{S}$ then bx_Z factors through $1 : 1 \to \mathbb{S}$ and

(iii) for all $x_Z : Z \to X$ if bx_Z factors through $0 : 1 \to \mathbb{S}$ then ax_Z factors through $0 : 1 \to \mathbb{S}$.

Proof. Immediate from the fact that the lattice structure of open/closed subobjects agrees with the order enrichment in the manner shown in the lemma. ■

As motivation for the next lemma it is worth first looking at some facts about the subobject classifier, Ω. Now Ω is the initial frame so for any other frame ΩZ, there is a unique frame homomorphism $\Omega!^Z : \Omega \to \Omega Z$; it is given by

$$i \longmapsto \bigvee^{\uparrow}_{\Omega Z} \{0_{\Omega Z}\} \cup \{1_{\Omega Z} \mid 1 \leq i\}.$$

From this it follows that for any function $p_\# : \Omega Z \to \Omega$, (and any $c \in \Omega Z$, $i \in \Omega$) the weakened Frobenius law,

$$p_\#(c) \wedge i \leq p_\#(c \wedge \Omega !^Z(i))$$

holds. If further $p_\#$ is a dcpo homomorphism then the weakened coFrobenius law,

$$p_\#(c \vee \Omega !^Z(i)) \leq p_\#(c) \vee i$$

holds. These facts follow since $i \leq j$ if and only if $i = 1$ implies $j = 1$, for truth values i, j in a topos (recall that $\Omega = P\{*\}$, i.e. the power set of the singleton set, and so these facts are just basic set theory). These properties are very particular to set theory, however they are also true in the abstract setting that we have with $\mathbb{S}$ in $\mathbf{C}$ taking the role of a 'subobject classifier' and natural transformations taking the role of dcpo homomorphisms.

Lemma 11 *If $p_\# : \mathbb{S}^Z \to \mathbb{S}$ is a morphism in $[\mathbf{C}^{op}, \mathbf{Set}]$ then*

$$\text{(i)}\ \sqcap_{\mathbb{S}} (p_\# \times 1) \sqsubseteq p_\# \sqcap_{\mathbb{S}^Z} (1 \times \mathbb{S}^{!^Z})$$

$$\text{(ii)}\ \ p_\# \sqcup_{\mathbb{S}^Z} (1 \times \mathbb{S}^{!^Z}) \sqsubseteq \sqcup_{\mathbb{S}}(p_\# \times 1)$$

The notation "#" in $p_\#$ is currently meaningless, but is consistent with notation later. This lemma is essentially the same as Proposition 3.11 in [Taylor 00]. However here, in contrast, there is no assumption that $\mathbb{S}^Z$ exists as an object of $\mathbf{C}$. Topologically this means that there is no assumption that Z is locally compact.

Proof. (i) Given any object W and any $a : W \times Z \to \mathbb{S}$, $b : W \to \mathbb{S}$, it needs to be established that

$$[\sqcap_{\mathbb{S}}(p_\# \times 1)]_W(a, b) \sqsubseteq_{\mathbf{C}(W,\mathbb{S})} [p_\# \sqcap_{\mathbb{S}^Z} (1 \times \mathbb{S}^{!^Z})]_W(a, b).$$

The left hand side is

$$W \overset{((p_\#)_W(a),b)}{\to} \mathbb{S} \times \mathbb{S} \overset{\sqcap_{\mathbb{S}}}{\to} \mathbb{S}$$

which (closed) classifies $[(p_\#)_W(a)]^*(1) \wedge_{Sub(W)} b^*(1)$. The right hand side is

$$(p_\#)_W(W \times Z \overset{(a,b\pi_1)}{\to} \mathbb{S} \times \mathbb{S} \overset{\sqcap_{\mathbb{S}}}{\to} \mathbb{S}).$$

So, by the previous lemma (part (i)), it is sufficient to prove that for any $x : X \to W$ if $\sqcap_{\mathbb{S}}((p_\#)_W(a), b)x = 1!^X$ then $(p_\#)_W(\sqcap_{\mathbb{S}}(a, b\pi_1))x = 1!^X$. Now if $\sqcap_{\mathbb{S}}((p_\#)_W(a), b)x = 1!^X$, then $x : X \to W$ factors through $(p_\#(a))^*(1)$ and $b^*(1)$, i.e. $[(p_\#)_W(a)]x = 1!^X$ and $bx = 1!^X$. By naturality of $p_\#$,

$$(p_\#)_W(\sqcap_{\mathbb{S}}(a, b\pi_1))x = (p_\#)_X(\sqcap_{\mathbb{S}}(a, b\pi_1)(x \times 1)).$$

But,

$$
\begin{array}{rl}
& X \times Z \stackrel{x\times 1}{\to} W \times Z \stackrel{(a,b\pi_1)}{\to} \mathbb{S}\times\mathbb{S} \stackrel{\sqcap_{\mathbb{S}}}{\to} \mathbb{S} \\
= & X \times Z \stackrel{(a(x\times 1),bx\pi_1)}{\to} \mathbb{S}\times\mathbb{S} \stackrel{\sqcap_{\mathbb{S}}}{\to} \mathbb{S} \\
= & X \times Z \stackrel{(a(x\times 1),1!^{X\times Z})}{\to} \mathbb{S}\times\mathbb{S} \stackrel{\sqcap_{\mathbb{S}}}{\to} \mathbb{S} \\
= & X \times Z \stackrel{a(x\times 1)}{\to} \mathbb{S}
\end{array}
$$

where the last line follows since $1!^{X\times Z}$ is the unit of the semilattice $\mathbf{C}(X\times Z,\mathbb{S})$. Therefore,

$$
\begin{array}{rcll}
(p_\#)_W(\sqcap_{\mathbb{S}}(a,b\pi_1))x & = & (p_\#)_X(a(x\times 1)) & \\
& = & [(p_\#)_W(a)]x & \text{(by naturality of } p_\#) \\
& = & 1!^X. &
\end{array}
$$

(ii) The proof is entirely order dual, using (iii) of the previous lemma. ■

7 Representation Theorem for Weak Triquotient Assignments

Weak triquotient assignments will be used to define triquotient surjections, the class of maps which we wish to show are of effective descent. Further the idea of a weak triquotient assignment is a useful stepping stone towards a discussion of proper and open maps. This is because both proper and open maps are maps with weak triquotient assignments and it is pullback stability of weak triquotient assignments that can then be used to recover the more familiar results ([Vermeulen 93], [JoyTie 84]) that proper and open maps are pullback stable.

The next lemma exploits the distributivity axiom required of $\mathbb{S}$ and will provide a single equation characterization for the definition of weak triquotient assignment to follow.

Lemma 12 *Given $p: Z \to Y$ and a natural transformation $p_\#: \mathbb{S}^Z \to \mathbb{S}^Y$ then*

$$
\begin{array}{ccc}
\mathbb{S}^Z\times\mathbb{S}^Z\times\mathbb{S}^Y & \xrightarrow{1\times 1\times\mathbb{S}^p} & \mathbb{S}^Z\times\mathbb{S}^Z\times\mathbb{S}^Z \\
\downarrow{\scriptstyle (\pi_1,\pi_3,\sqcap(\pi_1,\pi_2))} & & \downarrow{\scriptstyle \sqcap(1\times\sqcup)} \\
\mathbb{S}^Z\times\mathbb{S}^Y\times\mathbb{S}^Z & & \mathbb{S}^Z \\
\downarrow{\scriptstyle p_\#\times 1\times p_\#} & & \downarrow{\scriptstyle p_\#} \\
\mathbb{S}^Y\times\mathbb{S}^Y\times\mathbb{S}^Y & \xrightarrow{\sqcup(\sqcap\times 1)} & \mathbb{S}^Y
\end{array}
$$

commutes if and only if

$$\text{(a)} \qquad p_\#(c) \sqcap b \sqsubseteq p_\#(c \sqcap \mathbb{S}^p(b))$$

and

$$\text{(b)} \qquad p_\#(c \sqcup \mathbb{S}^p(b)) \sqsubseteq p_\#(c) \sqcup b.$$

Proof. Written lattice theoretically, the commuting diagram is saying

$$p_\#(c_1 \sqcap [c_2 \sqcup \mathbb{S}^p(b)]) = (p_\#(c_1) \sqcap b) \sqcup p_\#(c_1 \sqcap c_2) \qquad (*)$$

for all c_1, c_2"$\in$"$\mathbb{S}^Z$ and $\forall b$"$\in$"$\mathbb{S}^Y$ and this certainly implies (a) and (b) by taking $c_2 = 0$ to obtain (a) and $c_1 = 1$ to obtain (b).

Showing the converse is a straightforward exercise in the distributivity axiom. Assume (a) and (b). Certainly $p_\#(c_1 \sqcap c_2) \sqsubseteq p_\#(c_1 \sqcap [c_2 \sqcup \mathbb{S}^p(b)])$. Also

$$\begin{aligned} (p_\#(c_1) \sqcap b) &\sqsubseteq p_\#(c_1 \sqcap \mathbb{S}^p(b)) \\ &\sqsubseteq p_\#(c_1 \sqcap [c_2 \sqcup \mathbb{S}^p(b)]) \end{aligned}$$

and so RHS$\sqsubseteq$LHS for $(*)$. By distributivity the RHS of $(*)$ can be re-expressed as $p_\#(c_1) \sqcap [b \sqcup p_\#(c_1 \sqcap c_2)]$ (since $p_\#(c_1 \sqcap c_2) \sqsubseteq p_\#(c_1)$). But, $p_\#(c_1 \sqcap [c_2 \sqcup \mathbb{S}^p(b)]) \sqsubseteq p_\#(c_1)$ and so it remains only to verify that

$$p_\#(c_1 \sqcap [c_2 \sqcup \mathbb{S}^p(b)]) \sqsubseteq b \sqcup p_\#(c_1 \sqcap c_2).$$

This follows since,

$$\begin{aligned} p_\#(c_1 \sqcap [c_2 \sqcup \mathbb{S}^p(b)]) &= p_\#([c_1 \sqcap c_2] \sqcup [c_1 \sqcap \mathbb{S}^p(b)]) \\ &\sqsubseteq p_\#([c_1 \sqcap c_2] \sqcup \mathbb{S}^p(b)) \\ &\sqsubseteq p_\#(c_1 \sqcap c_2) \sqcup b. \end{aligned}$$

■

Definition 13 *If $p : Z \to Y$ is a morphism in* **C** *then a* weak triquotient assignment *on p is a natural transformation $p_\# : \mathbb{S}^Z \to \mathbb{S}^Y$ satisfying the conditions of the lemma.*

Lemma 11 of the previous section is therefore showing that every map $p_\# : \mathbb{S}^Z \to \mathbb{S}$ is a weak triquotient assignment on $! : Z \to 1$.

Weak triquotient assignments on locales were originally introduced as (very) weak triquotient assignments by Vickers in [Vickers 01a] using different, but equivalent, equations. They are generalizations of Plewe's localic triquotient assignments, [Plewe 97], but are strictly weaker since the existence of one of our weak triquotient assignments on p does not imply that p is a surjection. In fact every map has two trivial triquotient assignments.

Example 14 *If $p : Z \to Y$ is a map in* $\mathbf{C}$ *then*

$$\mathbb{S}^Z \to 1 \xrightarrow{0} \mathbb{S}^Y$$

and

$$\mathbb{S}^Z \to 1 \xrightarrow{1} \mathbb{S}^Y$$

are both weak triquotient assignments on p. This follows since both sides of the defining equation are, respectively, 0 *and* 1 *for each $c_1, c_2 \in \mathbb{S}^Z$ and $b \in \mathbb{S}^Y$.*

So, there can be no claim that weak triquotient assignments on p are unique as every map with non-trivial codomain has at least two triquotient assignments (they are bottom and top in the poset of all weak triquotient assignments).

Let us now introduce the non-trivial examples of weak triquotient assignments that will guide our applications.

Example 15 *[Michael 77] If $p : Z \to Y$ is a continuous map of topological spaces then p is called a* surjective triquotient map *if there exists a map $p_\# : Opens(Z) \to Opens(Y)$ such that*

(T1) $p_\#(U) \subseteq p(U)$

(T2) $p_\#(Z) = Y$

(T3) $p_\#(U) \subseteq p_\#(V)$ if $U \subseteq V$

(T4) for all $y \in p_\#(U)$ and each directed cover $\mathcal{W}$ of $p^{-1}(y) \cap U$ there exists $V \in \mathcal{W}$ such that $y \in p_\#(V)$.

Certainly $\mathcal{W}$ covers $p^{-1}(y) \cap \bigcup^\uparrow \mathcal{W}$ for any directed collection $\mathcal{W}$ and any y. But then $y \in p_\#(\bigcup^\uparrow \mathcal{W})$ implies there exists $V \in \mathcal{W}$ such that $y \in p_\#(V)$ from which we see that (T4) implies that $p_\# \bigcup^\uparrow \mathcal{W} \subseteq \bigcup^\uparrow_{V \in \mathcal{W}} p_\#(V)$ and so $p_\#$ is a dcpo homomorphism given, (T3), that it is monotone. (T4) implies $p_\#(U) \cap V \subseteq p_\#(U \cap p^{-1}(V))$ since $y \in V$ implies $\{U \cap p^{-1}(V)\}$ covers $p^{-1}(y) \cap U$. Further if $y \in p_\#(U \cup p^{-1}V)$ and $y \notin V$ then $\{U\}$ covers $p^{-1}(y) \cap (U \cup f^{-1}V)$ and so $p_\#(U \cup p^{-1}(V)) \subseteq p_\#(U) \cup V$. Therefore, with (T3) and (T4) we have

$$\begin{aligned} p_\#(U) \cap V &\subseteq p_\#(U \cap p^{-1}(V)) \\ p_\#(U \cup p^{-1}(V)) &\subseteq p_\#(U) \cup V \end{aligned}$$

which is sufficient to show that $p_\#$ is a weak triquotient assignment for $\mathbf{C} = \mathbf{Loc}$*, treating Z and Y as locales via their frames of open subsets. Examples of maps $p_\#$ that arise in topology are via sections ($p_\# = s^{-1}$ for a section s), open surjections ($p_\#$ =direct image) and proper surjections, i.e. closed surjections with compact fibres ($p_\#(U) = p(U^c)^c$).*

Example 16 *[Plewe 97] If $p : Z \to Y$ is a mapping of locales then p is a* triquotient surjection *iff there exists a* t-assignment *on p; that is a dcpo homomorphism $p_\# : \Omega X \to \Omega Y$ such that*

$$\begin{aligned} \text{(F1) } p_\#(c \wedge \Omega p(b)) &= p_\#(c) \wedge b \\ \text{(F2) } p_\#(c \vee \Omega p(b)) &= p_\#(c) \vee b. \end{aligned}$$

Recall that a topological space, Y, is a T_D space provided for every $y \in Y$ there exists an open U such that $\{y\} = cl\{y\} \cap U$, where $cl\{y\}$ is the closure of $\{y\}$. In other words a space is T_D if and only if every point is an open subspace of its closure. Say $p : Z \to Y$ is a continuous map between topological spaces and Y is T_D then, see [Plewe 97], the localic t-assignments on p, treating p as a locale map, are exactly the maps $Opens(Z) \to Opens(Y)$ satisfying (T1)-(T4). So p, treated as a locale map, is a triquotient surjection if and only if it is a topological triquotient surjection (provided Y is T_D).

Clearly with $\mathbf{C} = \mathbf{Loc}$ *every t-assignment is a weak triquotient assignment. It is easy to check that a weak triquotient assignment is a t-assignment if and only if $p_\# \Omega p = Id$, equivalently $p_\#(0) = 0$ and $p_\#(1) = 1$. Note that $p_\# \Omega p = Id$ forces Ωp to be a monomorphism, i.e. p is an epimorphism partly justifying the term 'surjection'. Below we prove that p is a regular epimorphism, thereby completely justifying the term 'surjection'.*

Example 17 *[JoyTie 84] A locale map $p : Z \to Y$ is* open *if*

(O1) there exists $\exists_p : \Omega Z \to \Omega Y$ left adjoint to $\Omega p : \Omega Y \to \Omega Z$ and

(O2) $\exists_p(c \wedge \Omega p(b)) = b \wedge \exists_p(c)$, for all $c \in \Omega Z, b \in \Omega Y$. (Frobenius.)

If Z and Y are topological spaces, then any topological open map is a localic open map; $\exists_p$ is the direct image function. As with our discussion of topological triquotient surjections, topological open maps coincide with localic open maps provided Y is T_D. $\exists_p$ is left adjoint and so preserves all joins, and so is a dcpo homomorphism. Further

$$\begin{aligned} \exists_p(c \vee \Omega p(b)) &= \exists_p(c) \vee \exists_p \Omega p(b) \\ &\leq \exists_p(c) \vee b \end{aligned}$$

and so $\exists_p$ is a weak triquotient assignment (with $\mathbf{C} = \mathbf{Loc}$*). Note that a weak triquotient assignment satisfies (O1) and (O2) if and only if it is left adjoint to Ωp.*

If, further, $p : Z \to Y$ is an epimorphism then Ωp is an inclusion and so since $\Omega p \exists_p \Omega p = \Omega p$ (as $\exists_p$ is left adjoint to Ωp) we have that $\exists_p \Omega p = Id$. So a locale map, p, is an open surjection *if and only if it is open and $\exists_p \Omega p = Id$. Note that given (O2) $\exists_p \Omega p = Id$ is equivalent to $\exists_p(1) = 1$.*

Example 18 *[Vermeulen 93] A locale map $p : Z \to Y$ is* proper *if*

(P1) the right adjoint to $\Omega p : \Omega Y \to \Omega Z$, denoted $\forall_p : \Omega Z \to \Omega Y$, is a dcpo homomorphism and

(P2) $\forall_p(c \vee \Omega p(b)) = b \vee \forall_p(c)$, for all $c \in \Omega Z, b \in \Omega Y$. (coFrobenius.)

If Z and Y are topological spaces, then any topological proper map is a localic proper map; $\forall_p(U) = (p(U^c))^c$. $\forall_p$ is right adjoint and so preserves meets. Therefore

$$\begin{aligned} \forall_p(c \wedge \Omega p(b)) &= \forall_p(c) \wedge \forall_p \Omega p(b) \\ &\geq \forall_p(c) \wedge b \end{aligned}$$

and so $\forall_p$ is a weak triquotient assignment (with $\mathbf{C}=\mathbf{Loc}$). Note that a weak triquotient assignment satisfies (P1) and (P2) if and only if it is right adjoint to Ωp.

A locale map, p, is a proper surjection *if and only if it is proper and $\forall_p \Omega p = Id$. Note that given (P2) $\forall_p \Omega p = Id$ is equivalent to $\forall_p(0)=0$.*

Given these examples, the following definitions using the axiomatic notion of weak triquotient assignment are well motivated and coincide with the usual definitions when $\mathbf{C}=\mathbf{Loc}$.

Definition 19 *A map $p: Z \to Y$ is said to be*

(a) a triquotient surjection *if there exists a weak triquotient assignment $p_{\#}$ such that $p_{\#}\mathbb{S}^p = Id$,*

(b) open *if there exists a weak triquotient assignment $p_{\#}$ left adjoint to $\mathbb{S}^p$,*

(c) proper *if there exists a weak triquotient assignment $p_{\#}$ right adjoint to $\mathbb{S}^p$,*

(d) an open surjection *if it is open and $p_{\#}\mathbb{S}^p = Id$ (equivalently, $p_{\#}(1)=1$) and*

(e) a proper surjection *if it is proper and $p_{\#}\mathbb{S}^p = Id$ (equivalently, $p_{\#}(0)=0$).*

Work contained in [Townsend 03] shows that weak triquotient assignments on a locale map $p: Z \to Y$ correspond exactly to dcpo maps $\Omega_{ShY} Z_p \to \Omega_{ShY}$. This representation theorem provided a method for showing pullback stability of weak triquotient assignments. Lemma 11 proves this representation result axiomatically for $Y=1$. In fact, by exploiting Axiom 5, we can prove this representation theorem for weak triquotient assignments axiomatically for every Y.

Theorem 20 *Given an object Z_p in $\mathbf{C}/Y$ (i.e. a map $p: Z \to Y$ in $\mathbf{C}$) there is a 1-1 correspondence between natural transformations $\mathbb{S}_Y^{Z_p} \to \mathbb{S}_Y$ and weak triquotient assignments on p.*

This theorem is the central technical observation of this paper. From it the pullback stability of weak triquotient assignments will be trivial, and all the other pullback stability results will follow. The proof exploits various aspects of the axiomatization simultaneously, e.g. the slice stability, the equalizer→coequalizer axiom (Axiom 5) and the classifying properties of the Sierpiński object (Axiom 3).

Proof. An overview of the proof is simple. For any $p: Z \to Y$ in $\mathbf{C}/Y$ there is an equalizer diagram

$$Z_p \overset{(p,1)}{\to} Z_Y \underset{\Delta\pi_2}{\overset{p\times 1}{\rightrightarrows}} Y_Y \qquad (*)$$

in $\mathbf{C}/Y$ where Z_Y denotes $\pi_1: Y \times Z \to Y$. So by Axiom 5, in $\mathbf{C}/Y$, maps $\mathbb{S}_Y^{Z_p} \to \mathbb{S}_Y$ are exactly maps $\mathbb{S}_Y^{Z_Y} \to \mathbb{S}_Y$ which coequalize the diagram $(*)$ when the contravariant functor $\mathbb{S}_Y^{(\text{-})} : \mathbf{C}/Y \quad \rightarrow \quad \mathbf{C}_Y^{op}$ is applied (and suitably join/meet closed, as required by Axiom 5). But Z_Y is the pullback of $!^Z : Z \rightarrow 1$ along $!^Y : Y \rightarrow 1$ and so by the adjunction $(!^Y)^{\#} \dashv !^Y_*$ it is clear that such maps are exactly maps $\mathbb{S}^Z \rightarrow !^Y_* \mathbb{S}_Y \cong \mathbb{S}^Y$ whose adjoint transposes compose equally with a pair of morphism in $\mathbf{C}_Y^{op}$ (determined by the Axiom 5). The detail of the proof is therefore to show that for a map $p_{\#} : \mathbb{S}^Z \rightarrow !^Y_* \mathbb{S}_Y \cong \mathbb{S}^Y$, $p_{\#}$ is a weak triquotient assignment for p if and only if its adjoint transpose under $(!^Y)^{\#} \dashv !^Y_*$ composes equally with the meet/join closure of the image of the fork $(*)$ under $\mathbb{S}_Y^{(\text{-})}$.

It must therefore be shown that if $p'_{\#} : \mathbb{S}_Y^{Z_Y} \rightarrow \mathbb{S}_Y$ is the adjoint transpose of $p_{\#}$, then $p_{\#}$ is a weak triquotient assignment iff $p'_{\#}$ coequalizes

$$\mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Y_Y} \underset{1 \times 1 \times (\mathbb{S}_Y^{\pi_2} \mathbb{S}_Y^{\Delta})}{\overset{1 \times 1 \times \mathbb{S}_Y^{p \times 1}}{\rightrightarrows}} \mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Z_Y} \xrightarrow{\sqcap(1 \times \sqcup)} \mathbb{S}_Y^{Z_Y} \qquad (**)$$

Now, $p_{\#}$ is a weak triquotient assignment iff the diagram

$$\begin{array}{ccc}
\mathbb{S}^Z \times \mathbb{S}^Z \times \mathbb{S}^Y & \xrightarrow{1 \times 1 \times \mathbb{S}^p} & \mathbb{S}^Z \times \mathbb{S}^Z \times \mathbb{S}^Z \\
{\scriptstyle (\pi_1, \pi_3, \sqcap(\pi_1, \pi_2))} \downarrow & & \downarrow {\scriptstyle \sqcap(1 \times \sqcup)} \\
\mathbb{S}^Z \times \mathbb{S}^Y \times \mathbb{S}^Z & & \mathbb{S}^Z \\
{\scriptstyle p_{\#} \times 1 \times p_{\#}} \downarrow & & \downarrow {\scriptstyle p_{\#}} \\
\mathbb{S}^Y \times \mathbb{S}^Y \times \mathbb{S}^Y & \xrightarrow{\sqcup(\sqcap \times 1)} & \mathbb{S}^Y
\end{array}$$

commutes. It has been observed already that the pullback of $\sqcap_{\mathbb{S}}$ (along $! : Y \rightarrow 1$) is $\sqcap_{\mathbb{S}_Y^{Z_Y}}$ (i.e. $\sqcap_{\mathbb{S}_Y^{Z_Y}} = (!^Y)^{\#} \sqcap_{\mathbb{S}^Z}$, Lemma 8) and clearly the pullback of $\mathbb{S}^p$ is $\mathbb{S}_Y^{p \times 1}$. Therefore the adjoint transpose of the top and right hand part of this diagram is equal to the top row of $(**)$ postcomposed with $p'_{\#}$. The proof will be completed provided it can be shown that the adjoint transpose of the left and bottom part of this diagram is equal to the bottom row of $(**)$ postcomposed with $p'_{\#}$. Since $\sqcup_{\mathbb{S}^Y} = !^Y_* (\sqcup_{\mathbb{S}_Y})$ (see Lemma 8) the proof amounts to showing that the diagram

$$\begin{array}{ccc}
\mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Y_Y} & \xrightarrow{1 \times 1 \times (\mathbb{S}_Y^{\pi_2} \mathbb{S}_Y^{\Delta})} & \mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Z_Y} \\
{\scriptstyle (\pi_1, \pi_3, \sqcap(\pi_1, \pi_2))} \downarrow & & \downarrow {\scriptstyle \sqcap(1 \times \sqcup)} \\
\mathbb{S}_Y^{Z_Y} \times \mathbb{S}_Y^{Y_Y} \times \mathbb{S}_Y^{Z_Y} & & \mathbb{S}_Y^{Z_Y} \\
{\scriptstyle p'_{\#} \times \mathbb{S}_Y^{\Delta} \times p'_{\#}} \downarrow & & \downarrow {\scriptstyle p'_{\#}} \\
\mathbb{S}_Y \times \mathbb{S}_Y \times \mathbb{S}_Y & \xrightarrow{\sqcup(\sqcap \times 1)} & \mathbb{S}_Y
\end{array}$$

commutes. Noting that $\mathbb{S}_Y^{\pi_2} = \mathbb{S}_Y^{!^{Z_Y}}$, this follows by an application of Lemma 11 in the slice category $\mathbf{C}/Y$. ■

Remark 21 *(Explicit description of representation bijection.) It is worth making the bijection of this theorem explicit. In one direction it takes any map* $\beta : \mathbb{S}_Y^{Z_p} \to \mathbb{S}_Y$*, precomposes with* $\mathbb{S}_Y^{(p,1)} : \mathbb{S}_Y^{Z_Y} \to \mathbb{S}_Y^{Z_p}$ *and then takes the adjoint transpose (via* $(!^Y)^{\#} \dashv !^Y_*$ *). But* $\mathbb{S}_Y^{(p,1)} : \mathbb{S}_Y^{Z_Y} \to \mathbb{S}_Y^{Z_p}$ *is the counit of the adjunction* $(!^Y)^{\#} \dashv !^Y_*$ *and so what this representation theorem is showing is that every triquotient assignment* $p_{\#}$ *(on* $p : Z \to Y$*) is given by* $p_{\#} = !^Y_*(\beta)$ *for some unique* $\beta : \mathbb{S}_Y^{Z_p} \to \mathbb{S}_Y$*.*

We now focus on some simple applications. The change of base result preserves composition and so we can recover an alternative description of proper maps familiar from locale theory: that is a locale map $p : Z \to Y$ is proper if and only if Z_p is compact as an object in $\mathbf{Loc}_{Sh(Y)}$ (e.g. [Johnstone 81]).

Definition 22 *An object* Z *in* $\mathbf{C}$ *is* compact *if* $!^Z : Z \to 1$ *is proper.*

For $\mathbf{C} = \mathbf{Loc}$ this is the usual definition since $!^Z : Z \to 1$ is proper if and only if the right adjoint $\forall_{!^Z} : \Omega Z \to \Omega$ (to $\Omega !^Z$) is a dcpo homomorphism. The coFrobenius condition (P2) is always true for the right adjoint since the codomain is 1. It is easy to verify that the usual definition of locale compactness (i.e. that if directed $T \subseteq^{\uparrow} \Omega X$ has $\bigvee^{\uparrow} T = 1_{\Omega X}$ then $1_{\Omega X} \in T$) is equivalent to the requirement that the right adjoint to $\Omega !^Z$ is a dcpo homomorphism.

Corollary 23 *An object* Z_p *in* $\mathbf{C}/Y$ *is compact if and only if* $p : Z \to Y$ *is proper.*

Proof. If Z_p is compact then there exists $\beta : \mathbb{S}_Y^{Z_p} \to \mathbb{S}_Y$ such that $\mathbb{S}_Y^{!^{Z_p}} \dashv \beta$. But $!^Y_*$ is order preserving and takes $\mathbb{S}_Y^{!^{Z_p}}$ to $\mathbb{S}^p$. Therefore p is proper.

On the other hand if $p : Z \to Y$ is proper then $\mathbb{S}^p \dashv p_\#$ where $p_\# = !^Y_*(\beta)$ for some unique $\beta : \mathbb{S}_Y^{Z_p} \to \mathbb{S}_Y$. Therefore

$$\begin{array}{rcll} Id & \sqsubseteq & !^Y_*(\beta)\mathbb{S}^p & \quad (a) \\ \mathbb{S}^p !^Y_*(\beta) & \sqsubseteq & Id. & \quad (b) \end{array}$$

The adjoint transpose of $\mathbb{S}^p : \mathbb{S}^Y \to !^Y_* \mathbb{S}_Y^{Z_p}$ is $\mathbb{S}_Y^{(p,p)}$ and so since adjoint transpose preserves order we have that

$$\mathbb{S}_Y^{\Delta} \sqsubseteq \beta \mathbb{S}_Y^{(p,p)}$$

from (a) and so

$$Id \sqsubseteq \beta \mathbb{S}_Y^{!^{Z_p}}$$

by precomposing with $\mathbb{S}_Y^{\pi_2}$.

Now the adjoint transpose of (b) implies

$$\mathbb{S}_Y^{(p,p)}(!^Y)^\# !^Y_*(\beta) \sqsubseteq \mathbb{S}_Y^{(p,1)}$$

and so

$$\mathbb{S}_Y^{!^{Z_p}} \beta \mathbb{S}_Y^{(p,1)} \sqsubseteq \mathbb{S}_Y^{(p,1)}$$

since $\mathbb{S}_Y^{(p,p)}$ factors as $\mathbb{S}_Y^{!^{Z_p}} \mathbb{S}_Y^{\Delta}$ and $\mathbb{S}_Y^{\Delta}(!^Y)^\# !^Y_*(\beta)$ factors as $\beta \mathbb{S}_Y^{(p,1)}$, the latter by naturality of the counit. Now $\mathbb{S}_Y^{(p,1)}$ is, via Axiom 5, a coequalizer, but it can also be verified to be an order enriched coequalizer since the homsets are lattices and meets (or joins) commute with morphism composition. Therefore

$$\mathbb{S}_Y^{!^{Z_p}} \beta \sqsubseteq Id.$$

This is sufficient to prove that $!^{Z_p}$ is proper (relative to $\mathbf{C}/Y$) since any $\beta : \mathbb{S}_Y^{Z_p} \to \mathbb{S}_Y$ is a triquotient assignment for $!^{Z_p}$ by Lemma 11 carried out relative to $\mathbf{C}/Y$. Therefore Z_p is compact as an object of $\mathbf{C}/Y$. ■

Using exactly the same proof, but replacing the inequality $\sqsubseteq$ with equality, we have a new description of triquotient surjections:

Corollary 24 *$p : Z \to Y$ is a triquotient surjection if and only if there exists $\beta : \mathbb{S}_Y^{Z_p} \to \mathbb{S}_Y$ such that $\beta \mathbb{S}_Y^{!^{Z_p}} = Id$.*

Proof. If there exists such β then $!^Y_*(\beta)$ is a triquotient assignment on p which splits $\mathbb{S}^p$ and so p is a triquotient surjection. In the other direction if p is a triquotient surjection then $Id = !^Y_*(\beta)\mathbb{S}^p$ for some $\beta : \mathbb{S}_Y^{Z_p} \to \mathbb{S}_Y$. Then, as in the last proof, $Id = \beta \mathbb{S}_Y^{!^{Z_p}}$ by taking adjoint transpose and precomposing with, say, $\mathbb{S}_Y^{\pi_2}$. ■

Further in the last but one proof 'right adjoint' could be replaced with 'left adjoint' throughout, and so there is an identical conclusion for open maps.

Let us move to the 'open side' of this proper/open parallel in order to verify, axiomatically, the well known sheaf theoretic fact

$$\mathbf{LH}/Y \simeq Sh(Y);$$

that is, the category of sheaves on a topological space Y is equivalent to the category of local homeomorphisms over that space. For any topological space Y, $Sh(Y)$ can be embedded in $\mathbf{Loc}_{Sh(Y)}$ as the category of discrete locales. This follows since the property of being discrete can be determined in terms of open maps:

Definition 25 *An object Z in $\mathbf{C}$ is* discrete *if $!^Z : Z \to 1$ and $\Delta : Z \hookrightarrow Z \times Z$ are both open.*

[JoyTie 84] shows that a locale Z has $!^Z : Z \to 1$ and $\Delta : Z \hookrightarrow Z \times Z$ open if and only if $\Omega Z \cong P(A)$ for some set A and in this way **Set** embeds into **Loc** as the full subcategory of discrete locales. The argument is constructive and so carried out relative to sheaves over Y, $Sh(Y)$ can be embedded in $\mathbf{Loc}_{Sh(Y)}$.

Definition 26 *A map $p : Z \to Y$ in $\mathbf{C}$ is a* local homeomorphism *if both p and the diagonal $\Delta : Z \hookrightarrow Z \times_Y Z$ are open.*

It is known (e.g. C1.3 [Johnstone 02]) that if Y is a topological space, then every locale map $p : Z \to Y$ which is a local homeormorphism in the sense just defined ($\mathbf{C} = \mathbf{Loc}$) arises from a unique topological local homeomorphism. We omit the details in proving this well known fact, but provided Y is a topological space then the category of local homeomorphisms over Y, denoted $\mathbf{LH}/Y$, is the same category, up to equivalence, viewed either as a full subcategory of $\mathbf{Loc}/Y$ or as a full subcategory of $\mathbf{Top}/Y$.

Corollary 27 *For any Y in $\mathbf{C}$, $\mathbf{LH}/Y \cong \mathbf{DisLoc}_Y$ where $\mathbf{LH}/Y$ is the full subcategory of $\mathbf{C}/Y$ whose objects are local homeomorphisms and $\mathbf{DisLoc}_Y$ is the full subcategory of $\mathbf{C}/Y$ whose objects are discrete relative to Y.*

Proof. Reversing the order in the proof of the previous corollary we have that $p : Z \to Y$ is open if and only if $!^{Z_p} : Z_p \to 1$ is open relative to Y. It remains therefore only to check that the map $\Delta : Z \hookrightarrow Z \times_Y Z$ is open (relative to 1) if and only if it is open relative to Y.

Now if $\Delta_\# : \mathbb{S}_Y^{Z_p} \to \mathbb{S}_Y^{Z \times_Y Z}$ is a triquotient assignment left adjoint to $\mathbb{S}_Y^{\Delta} : \mathbb{S}_Y^{Z_p} \to \mathbb{S}_Y^{Z \times_Y Z}$, then $!_*^Y(\Delta_\#)$ is a weak triquotient assignment left adjoint to $!_*^Y(\mathbb{S}_Y^{\Delta})$ since $!_*^Y$ preserves order and the lattice structure. Therefore Δ is open since $!_*^Y(\mathbb{S}_Y^{\Delta}) = \mathbb{S}^{\Delta}$ and so $\mathbf{DisLoc}_Y \subseteq \mathbf{LH}/Y$.

Finally, say $\Delta : Z \hookrightarrow Z \times_Y Z$ is open, then, arguing relative to $Z \times_Y Z$ we have that $!^{Z_\Delta} : Z_\Delta \to 1$ is open, from which there exists a weak triquotient assignment,

$$\beta : \mathbb{S}_{Z \times_Y Z}^{Z_\Delta} \to \mathbb{S}_{Z \times_Y Z}$$

left adjoint to $\mathbb{S}^{!^{Z_\Delta}}_{Z\times_Y Z} : \mathbb{S}^{Z_\Delta}_{Z\times_Y Z} \to \mathbb{S}_{Z\times_Y Z}$. But then by change of base back to $\mathbb{S}_Y$, via $p\pi_1$ say, we have that $(p\pi_1)_*(\beta)$ is a weak triquotient assignment left adjoint to $\mathbb{S}^{\Delta}_Y$ since $(p\pi_1)_*(\mathbb{S}^{!^{Z_\Delta}}_{Z\times_Y Z}) = \mathbb{S}^{\Delta}_Y$. ■

The following table summarizes the various definitions of classes of maps in terms of weak triquotient assignments $p_\#$.

Class	**Conditions on w.t.a.** $p_\#$	**Equivalent Conditions**	**Cond. on** $\beta : S^{Z_p}_Y \to S_Y$
Triq. Surj.	$p_\#\mathbb{S}^p = Id$	$p_\#(0) = 0$ & $p_\#(1) = 1$	$\beta\mathbb{S}^{!^{Z_p}}_Y = Id$
Proper	$\mathbb{S}^p \dashv p_\#$		$\mathbb{S}^{!^{Z_p}}_Y \dashv \beta$
Open	$p_\# \dashv \mathbb{S}^p$		$\beta \dashv \mathbb{S}^{!^{Z_p}}_Y$
Proper Surj.	Proper & $p_\#\mathbb{S}^p = Id$	Proper & $p_\#(0) = 0$	$\beta\mathbb{S}^{!^{Z_p}}_Y = Id$ & $\mathbb{S}^{!^{Z_p}}_Y\beta \sqsubseteq Id$
Open Surj.	Open & $p_\#\mathbb{S}^p = Id$	Open & $p_\#(1) = 1$	$\beta\mathbb{S}^{!^{Z_p}}_Y = Id$ & $Id \sqsubseteq \mathbb{S}^{!^{Z_p}}_Y\beta$

'w.t.a' stands for weak triquotient assignment. The last column gives an equivalent characterization of the class of map in terms of a natural transformation, β, relative to Y.

8 Pullback stability of Maps with Weak Triquotient Assignments

With the description of weak triquotient assignments on p just given, a general pullback stability result is straightforward.

Theorem 28 *If $p : Z \to Y$ is a map with a weak triquotient assignment $p_\# : \mathbb{S}^Z \to \mathbb{S}^Y$ and $f : X \to Y$ is any map in* **C** *then the pullback $f^*p : X \times_Y Z \to X$ has a unique weak triquotient assignment denoted $(f^*p)_\#$ such that the Beck-Chevalley condition holds, that is*

$$\begin{array}{ccc} \mathbb{S}^{X\times_Y Z} & \overset{\mathbb{S}^{p^*f}}{\longleftarrow} & \mathbb{S}^Z \\ (f^*p)_\# \downarrow & & \downarrow p_\# \\ \mathbb{S}^X & \overset{\mathbb{S}^f}{\longleftarrow} & \mathbb{S}^Y \end{array}$$

commutes.

This was originally shown by Vickers for the category of locales in [Vickers 01a].

Proof. Set $(f^*p)_\# = !^Y_*(f^\#(\beta))$, where $\beta : \mathbb{S}^{Z_p}_Y \to \mathbb{S}_Y$ is unique such that $p_\# = !^Y_*(\beta)$. We have discussed already that $(f^*p)_\#$, so defined, is a weak triquotient assignment for $f^*p : X \times_Y Z \to X$ since any weak triquotient assignment on f^*p is uniquely of this form. The square

$$\begin{array}{ccc} \mathbb{S}_Y^{Z_p} & \stackrel{\overline{\eta}_{\mathbb{S}_Y^{Z_p}}}{\rightarrow} & f_* f^\# \mathbb{S}_Y^{Z_p} \\ \beta \downarrow & & \downarrow f_* f^\#(\beta) \\ \mathbb{S}_Y & \stackrel{\overline{\eta}_{\mathbb{S}_Y}}{\rightarrow} & f_* f^\# \mathbb{S}_Y \end{array}$$

commutes by naturality of the unit. It has been observed above (Lemma 8) that $!_*^Y(\overline{\eta}_{\mathbb{S}_Y^{Z_p}}) = \mathbb{S}^{p^*f} : \mathbb{S}^Z \to \mathbb{S}^{X \times_Y Z}$ and so the Beck-Chevalley condition follows by applying $!_*^Y$ to this square.

For uniqueness let $i : X \times_Y Z \hookrightarrow X \times Z$ be the inclusion. This map is a regular monomorphism and so $\mathbb{S}^i = !_*^Y(\mathbb{S}_X^i) : \mathbb{S}^{X \times Z} \to \mathbb{S}^{f^* Z_p}$ is an epimorphism in $\mathbf{C}_1^{op}$ by Axiom 5. Note that

$$\mathbb{S}^{p^*f} = \mathbb{S}^i \mathbb{S}^{\pi_2} \qquad \text{(a)}$$

where $\pi_2 : X \times Z \to Z$ and recall that $\mathbb{S}^{\pi_2} = \overline{\eta}_{\mathbb{S}^Z}$, i.e. the unit of the adjunction $(!^X)^\# \dashv !_*^X$ evaluated at $\mathbb{S}^Z$. Any triquotient assignment on $f^*p : X \times_Y Z \to X$ is uniquely of the form $!_*^X(\gamma)$ for some $\gamma : \mathbb{S}_X^{f^* Z_p} \to \mathbb{S}_X$ and so it remains to show that if

$$!_*^X(\delta_j)\mathbb{S}^{p^*f} = \mathbb{S}^f p_\# \qquad (*)$$

for $j = 1, 2$ then $!_*^X(\delta_1) = !_*^X(\delta_2)$. But the adjoint transpose of $!_*^X(\delta_j)\mathbb{S}^{p^*f}$ is $\delta_j \mathbb{S}_X^i$ since $\mathbb{S}^{\pi_2}$ is the unit and $\mathbb{S}^i = !_*^X(\mathbb{S}_X^i)$. Therefore $(*)$ implies that $\delta_1 \mathbb{S}_X^i = \delta_2 \mathbb{S}_X^i$, hence $!_*^X(\delta_1)\mathbb{S}^i = !_*^X(\delta_2)\mathbb{S}^i$ and so uniqueness follows since $\mathbb{S}^i$ is an epimorphism. ■

As a corollary we obtain all of the more familiar results:

Corollary 29 *The classes of triquotient surjections, proper maps, opens maps, proper surjections and open surjections are all pullback stable.*

Proof. Change of base respects order and so this is immediate from our characterization of these maps given above. For example, if p is a triquotient surjection then there exists there exists $\beta : \mathbb{S}_Y^{Z_p} \to \mathbb{S}_Y$ such that $\beta \mathbb{S}_Y^{!^{Z_p}} = Id$, from which $(f_*\beta)\mathbb{S}_Y^{!^{f^*Z_p}} = f_* Id = Id$ for any change of base map $f : X \to Y$ which is sufficient to prove that f^*p a triquotient surjection. Alternatively, for the surjections, the characterization in terms of $p_\#(0) = 0$ and/or $p_\#(1) = 1$ is pullback stable given the Beck-Chevalley condition proved on the pullback square. ■

We are now in a position to be able to verify that the definition of triquotient surjection is sufficient for p to be a regular epimorphism (and so justify the term 'surjection').

Lemma 30 *Any triquotient surjection is the coequalizer of its kernel pair.*

Proof. Say $p : Y \to Z$ is a triquotient surjection and $p_1, p_2 : Y \times_Z Y \to Y$ is the pullback of p against itself. Let $d : Y \twoheadrightarrow Q$ be the coequalizer of p_1, p_2 then d is a regular epimorphism and there exists a unique $f : Q \to Z$ such that $fd = p$. We prove that f^{-1} exists by checking that $\mathbb{S}^f$ is an isomorphism and appealing to Axiom 4. We claim that $p_\# \mathbb{S}^d$ is the inverse. Certainly $p_\# \mathbb{S}^d \mathbb{S}^f = p_\# \mathbb{S}^p = 1$ since $p_\#$ is the split of $\mathbb{S}^p$. But

$$\begin{aligned} \mathbb{S}^d \mathbb{S}^f p_\# \mathbb{S}^d &= \mathbb{S}^p p_\# \mathbb{S}^d \\ &= (p_1)_\# \mathbb{S}^{p_2} \mathbb{S}^d \\ &= (p_1)_\# \mathbb{S}^{p_1} \mathbb{S}^d = \mathbb{S}^d \end{aligned}$$

by, respectively, (i) the fact that $p = fd$, (ii) the Beck-Chevalley condition for the pullback, (iii) the fact that $dp_1 = dp_2$ and (iv) the fact that p_1 is a triquotient surjection. But $\mathbb{S}^d$ must be a monomorphism by Axiom 4 and so $\mathbb{S}^f$ is an isomorphism thereby completing the proof. ∎

This last proof is just a reinterpretation of Ch.V, Section 4, Prop. 2 in [JoyTie 84], which shows that open surjections between locales are regular epimorphisms. It of course recovers the fact that both proper and open surjections are coequalizers. For proper and open surjections this goes the other way:

Lemma 31 *If $p : Y \to Z$ is a regular epimorphism and proper/open then it is proper/open surjection.*

Proof. Since $p_\#$ is right (or left) adjoint to $\mathbb{S}^p$ we have that $\mathbb{S}^p p_\# \mathbb{S}^p = \mathbb{S}^p$ from which $p_\# \mathbb{S}^p = Id$ since (Axiom 4) $\mathbb{S}^p$ is a monomorphism. I.e. p a proper/open surjection . ∎

Thus a proper/open map is a surjection if and only if it is a regular epimorphism. It is not the case, axiomatically, that every regular epimorphism is a triquotient surjection since below an example is given of an effective descent morphism that is not a triquotient surjection. There are also non-pullback stable regular epimorphisms in **Loc** which therefore also witness this fact, see the Introduction in [Plewe 97].

For use in the next section, here is are some stability results.

Lemma 32 *(a) Given a cosplit equalizer diagram*

$$\mathbb{S}_Z^C \underset{b_\#}{\overset{\mathbb{S}_Z^b}{\rightleftarrows}} \mathbb{S}_Z^B \underset{(a_1)_\#}{\overset{\mathbb{S}_Z^{a_1}}{\underset{\mathbb{S}_Z^{a_2}}{\rightrightarrows}}} \mathbb{S}_Z^A$$

in $\mathbf{C}_Z^{op}$ then $A \underset{a_2}{\overset{a_1}{\rightrightarrows}} B \overset{b}{\twoheadrightarrow} C$ is a pullback stable coequalizer in $\mathbf{C}/Z$.

(b) Given a commuting diagram in $\mathbf{C}$

$$\begin{array}{ccc} X & \overset{g}{\to} & Y \\ & h \searrow & \downarrow f \\ & & Z \end{array}$$

with h a triquotient surjection then f is a triquotient surjection.

Proof. (a) The cosplit equalizer diagram is stable under pullback (i.e. under $p^{\#}$ for any $p : X \to Z$) and so it is sufficient to verify that such a diagram forces

$$A \underset{a_2}{\overset{a_1}{\rightrightarrows}} B \overset{b}{\twoheadrightarrow} C$$

to be a coequalizer in $\mathbf{C}/Z$. Certainly $\mathbb{S}_Z^{a_1}\mathbb{S}_Z^{b} = \mathbb{S}_Z^{a_2}\mathbb{S}_Z^{b}$ forces $ba_1 = ba_2$ since $\mathbb{S}_Z^{(\_)}$ is faithful. The remainder is shown exactly as in the proof of the previous lemma: take $d : B \twoheadrightarrow Q$ to be the coequalizer of a_1 and a_2, then $b = fd$ for some unique $f : Q \to C$ and it can be shown that $b_{\#}\mathbb{S}_Z^{d}$ is an inverse for $\mathbb{S}_Z^{f}$ and so f^{-1} exists.

(b) Take $f_{\#} = h_{\#}\mathbb{S}^{g}$. ■

9 Effective Descent of Triquotient Surjections

Recall, for example (3.1) in [JanSob 02], that a morphism $p : X \to Z$ in a cartesian category $\mathbf{C}$ is an *effective descent morphism* if the functor $p^* : \mathbf{C}/Z \to \mathbf{C}/X$ is monadic. That is, equivalently by definition, if and only if the comparison functor $\mathbf{C}/Z \to (\mathbf{C}/X)^{\mathbb{T}}$ is part of an equivalence of categories, where $\mathbb{T}$ is the monad induced by the adjunction $\Sigma_p \dashv p^*$. The problem of establishing which continuous maps are of effective descent in the category of topological spaces has been studied by a number of authors, for example [Michael 77] and [ReitThol 94]. A proof that open locale surjections are effective descent morphism was key to Joyal and Tierney's proof that every Grothendieck topos is equivalent to the category of continuous actions of a localic groupoid, [JoyTie 84]. There is therefore a natural interest in effective descent morphisms and it is Plewe's account of descent via triquotient assignments that is the most general available for locales [Plewe 97]. Moerdijk in [Moerdijk 89] develops an axiomatic account of descent, but as pointed out by Plewe in comments after Proposition 4.3 in [Plewe 97], these axioms are not satisfied by the class of localic triquotient surjections. The axiomatic account of effective descent offered here is therefore distinct from Moerdijk's (it is not more general since Moerdijk's account requires less of the ambient category $\mathbf{C}$).

We now embark on a proof of effective descent for triquotient surjections. The techniques of the proof are not new, they are taken from Section 4 of [Plewe 97]. The proof has been simplified slightly by arguing relative to the codomain of the triquotient surjection.

Via Beck's monadicity theorem (e.g. [MacLane 71]), given a category $\mathcal{C}$ with coequalizers, we have that a functor $U : \mathcal{C} \to \mathcal{D}$ is monadic iff U has a left adjoint, reflects isomorphisms and preserves coequalizers of U-split pairs of arrows (i.e. pairs $a, b : X \to Y$ such that $U(a), U(b) : UX \to UY$ has a split coequalizer). f^* reflects isomorphisms if f is a pullback stable regular epimorphism. This has just been established for triquotient surjections $p : X \twoheadrightarrow Z$ in the last section.

Therefore to prove that triquotient surjections are effective descent morphisms it remains to show that given a coequalizer

$$A \overset{a_1}{\underset{a_2}{\rightrightarrows}} B \overset{b}{\twoheadrightarrow} C$$

in $\mathbf{C}/Z$ with the property that the coequalizer

$$A \times_Z X \overset{a_1\times 1}{\underset{a_2\times 1}{\rightrightarrows}} B \times_Z X \overset{q}{\twoheadrightarrow} Q$$

is split in $\mathbf{C}/X$, we must have that

$$A \times_Z X \overset{a_1\times 1}{\underset{a_2\times 1}{\rightrightarrows}} B \times_Z X \overset{b\times 1}{\twoheadrightarrow} C \times_Z X$$

is a coequalizer in $\mathbf{C}/X$. Let $h : Q \to C \times_Z X$ by the unique map such that $hq = b \times 1$.

Now if this second coequalizer is split with, say $i : Q \to B \times_Z X$ and $j : B \times_Z X \to A \times_Z X$ satisfying $qi = 1_Q$, $(a_1 \times 1)j = 1_{B\times_Z X}$ and $(a_2 \times 1)j = iq$ then we have that both q and $a_1 \times 1$ are triquotient surjections with $q_\# \equiv \mathbb{S}^i_Z$ and $(a_1 \times 1)_\# \equiv \mathbb{S}^j_Z$. The argument to follow is going to be relative to Z.

By assumption $p : X \twoheadrightarrow Z$ is a triquotient surjection and so by changing base to Z we have that $!^{X_p} : X_p \to 1_Z$ is a triquotient surjection. Pulling back along $C \to Z$ and then along $B \overset{b}{\twoheadrightarrow} C$, we have that $\pi^C_1 : C \times_Z X \to C$ and $\pi^B_1 : B \times_Z X \to B$ are triquotient surjection in $\mathbf{C}/Z$ with assignments, say

$$\begin{aligned} (\pi^C_1)_\# &: \mathbb{S}^{C\times_Z X}_Z \to \mathbb{S}^C_Z \text{ and} \\ (\pi^B_1)_\# &: \mathbb{S}^{A\times_Z X}_Z \to \mathbb{S}^B_Z. \end{aligned}$$

By further pulling back we find that $\pi^A_1 : A\times_Z X \to A$ is a triquotient surjection and there exists $(\pi^A_1)_\# : \mathbb{S}^{A\times_Z X}_Z \to \mathbb{S}^A_Z$ such that $\mathbb{S}^c_Z(\pi^C_1)_\# = (\pi^A_1)_\#\mathbb{S}^{c\times 1}_Z$ where $c = ba_1 = ba_2$.

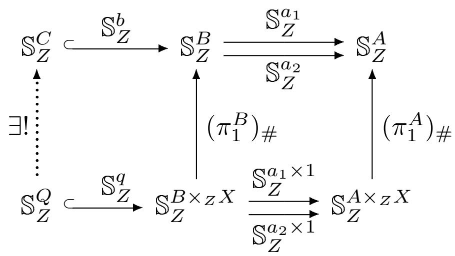
However $(\pi^A_1)_\#$ could have been obtained a different way; for $i = 1, 2$ let $(\pi^A_1)^i_\#$ be a weak triquotient assignment on $\pi^A_1 : A \times_Z X \to A$ such that $\mathbb{S}^{a_i}_Z(\pi^B_1)_\# = (\pi^A_1)_\#\mathbb{S}^{a_i\times 1}_Z$. Now, for $i = 1, 2$

$$\begin{aligned} (\pi^A_1)^i_\#\mathbb{S}^{c\times 1}_Z &= (\pi^A_1)^i_\#\mathbb{S}^{a_i\times 1}_Z\mathbb{S}^{b\times 1}_Z \\ &= \mathbb{S}^{a_i}_Z(\pi^B_1)_\#\mathbb{S}^{b\times 1}_Z \\ &= \mathbb{S}^{a_i}_Z\mathbb{S}^b_Z(\pi^C_1)_\# \\ &= \mathbb{S}^c_Z(\pi^C_1)_\# \end{aligned}$$

and so $(\pi^A_1)_\# = (\pi^A_1)^i_\#$ for $i = 1, 2$ by uniqueness of weak triquotient assignments satisfying Beck-Chevalley. Hence it does not matter how the weak triquotient assignment on π_1^A is obtained. Hence

$$\begin{aligned} \mathbb{S}_Z^{a_1}(\pi_1^B)_\# \mathbb{S}_Z^q &= (\pi_1^A)_\# \mathbb{S}_Z^{a_1 \times 1} \mathbb{S}_Z^q \\ &= (\pi_1^A)_\# \mathbb{S}_Z^{a_2 \times 1} \mathbb{S}_Z^q \\ &= \mathbb{S}_Z^{a_2}(\pi_1^B)_\# \mathbb{S}_Z^q. \end{aligned}$$

But $\mathbb{S}_Z^b$ is the equalizer $\mathbb{S}_Z^{a_1}, \mathbb{S}_Z^{a_2}$ by Axiom 4 and so there exists unique $(\pi_1^C h)_\# : \mathbb{S}_Z^Q \to \mathbb{S}_Z^C$ such that $(\pi_1^B)_\# \mathbb{S}_Z^q = \mathbb{S}_Z^b (\pi_1^C h)_\#$. Written out as a diagram this step is:



It is routine to verify that $(\pi_1^C h)_\#$ is a weak triquotient assignment on $\pi_1^C h$ satisfying $(\pi_1^C h)_\#(1) = 1$ and $(\pi_1^C h)_\#(0) = 0$. $\pi_1^C h$ is therefore a triquotient surjection so by stability of triquotient surjections under composition we have that $\pi_1^C hq = b\pi_1^B$ is a triquotient surjection. By the last lemma of the previous section we have that b is a triquotient surjection. Notice, from the proof of that lemma, $b_\# = (\pi_1^C h)_\# q_\# \mathbb{S}_Z^{\pi_1^B}$.

Recall that from construction of $q_\#$ and $(a_1 \times 1)_\#$, we have that $(a_1 \times 1)_\# \mathbb{S}_Z^{a_2 \times 1} = \mathbb{S}_Z^q q_\#$. Therefore for any a_1, a_2 with $a_1 \times 1, a_2 \times 1$ p^*-split we have (taking $(a_1)_\# = (\pi_1^B)_\# (a_1 \times 1)_\# \mathbb{S}^{\pi_1^A}$)

$$(a_1)_\# \mathbb{S}_Z^{a_2} = \mathbb{S}_Z^b b_\#.$$

which is sufficient, by (b) of the last lemma of the previous section to prove that the coequalizer

$$A \underset{a_2}{\overset{a_1}{\rightrightarrows}} B \overset{b}{\twoheadrightarrow} C$$

is pullback stable as required.

Therefore:

Theorem 33 *Triquotient surjections are effective descent morphisms in* **C**.

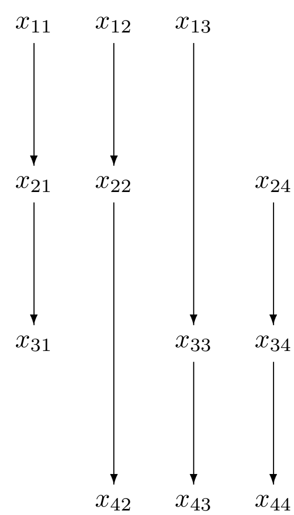
10 Effective Descent Morphism that is not a Triquotient Surjection

It is known for finite topological spaces that not every effective descent morphism is a triquotient surjection. See the comments after Proposition 7.1. [JanSob 02]. We now use their proof to show that:

Proposition 34 *The axioms do not imply that every effective descent morphism is a triquotient surjection.*

Proof. The category, **FinPos**, of finite posets is a model for the axioms and we now construct an effective descent morphism in **FinPos** which is not a triquotient surjection.

Effective descent morphisms between preorders are characterized in [JanSob 02]; they are exactly maps $p : X \to Z$ such that for every $z_1 \leq z_2 \leq z_3$ in Z there exists $x_1 \leq x_2 \leq x_3$ in X with the property $p(x_i) = z_i$ for $i = 1, 2, 3$. But the proof contained in that paper clearly also applies to the category of finite posets, and so a monotone map $p : X \to Z$ between finite posets is an effective descent morphism in **FinPos** provided every three chain is reflected. For example take X to be the reflexive, transitive closure of



take $Z = \{z_1 \leq z_2 \leq z_3 \leq z_4\}$ and define $p : X \to Z$ by $p(x_{ij}) = z_i$. Then p is an effective descent morphism in **FinPos**.

Now if p were an axiomatic triquotient surjection then there would exist a natural transformation $p_\# : \mathbb{S}^X \to \mathbb{S}^Z$ which is a weak triquotient assignment satisfying $p_\#(0) = 0$ and $p_\#(1) = 1$. But since **FinPos** is cartesian closed and $\mathbb{S} = \{0 \leq 1\}$, this means that there is a monotone map $p_\# : Open(X) \to Open(Z)$ where the opens are the Alexandrov opens (the upper closed subsets). This is sufficient to prove that p is a topological triquotient map in the sense of Michael, Example 15, since given any map $p : X \to Z$ between topological spaces we have that $p_\# : Open(X) \to Open(Z)$ is a topological triquotient assignment if it satisfies the axiomatic conditions for being a weak triquotient assignment (with $p_\#(0) = 0$ and $p_\#(1) = 1$) and Z is T_D. It is routine to verify that Z is T_D.

Now Michael's condition (T4) applied to finite posets amounts to

- if U and V are open subsets of X and $z \in Z$, then

$$z \in p_\#(U),\ p^{-1}(z) \cap U \subseteq V \ \Rightarrow\ z \in p_\#(V). \qquad (*)$$

Take $V = \uparrow p^{-1}(z_1)$, $U = X$ and $z = z_1$ in this to obtain $z_1 \in p_\#(\uparrow p^{-1}(z_1))$. But $p_\#(\uparrow p^{-1}(z_1))$ is open and so upper closed. Therefore $z_2 \in p_\#(\uparrow p^{-1}(z_1))$.

Now take $V = \uparrow (p^{-1}(z_2) \cap \uparrow p^{-1}(z_1))$, $U = \uparrow p^{-1}(z_1)$ and $z = z_2$ in $(*)$ to obtain $z_2 \in p_\#(V)$, and so

$$z_3 \in p[\uparrow (p^{-1}(z_2) \cap \uparrow p^{-1}(z_1))]$$

since $p_\#(V)$ is open an so upper closed and $p_\#(\_) \subseteq p(\_)$ by (T1). From which there exists $x''' \in \uparrow (p^{-1}(z_2) \cap \uparrow p^{-1}(z_1))$, $x''' \geq x'' \in p^{-1}(z_2) \cap \uparrow p^{-1}(z_1)$ and $x'' \geq x'$ with $p(x''') = z_3$, $p(x'') = z_2$ and $p(x') = z_1$. But this process can be repeated inductively and so we can obtain Prop. 7.1 of [JanSob 02] which states that if $p : X \to Z$ is a triquotient surjection between finite topological spaces then for every n, and every $z_1 \leq z_2 \leq ... \leq z_n$ in Z there exists $x_1 \leq x_2 \leq ... \leq x_n$ in X such that $p(x_i) = z_i$ for all i. This is not satisfied at $n = 4$ by the construction of our $p : X \to Z$ and so p is not an axiomatic triquotient surjection. ■

11 Triquotient Inclusions

In this section we look at triquotient inclusions, and show that the closed and open subobjects are examples. Following Plewe we give a complete characterization of triquotient inclusions: they are all the regular monomorphisms formed by the intersection of a closed subobject with an open subobject.

Definition 35 *A map $p : Z \to X$ is a triquotient inclusion if there exists a triquotient assignment $p_\#$ on p such that $\mathbb{S}^p p_\# = Id$.*

Just as with triquotient surjections we can adopt the proof of Lemma 7 (replacing inequalities with equalities) and show that $p : Z \to Y$ is a triquotient inclusion if and only if there exists $\beta : \mathbb{S}_Y^{Z_p} \to \mathbb{S}_Y$ such that $\mathbb{S}_Y^{!_{Z_p}} \beta = Id$ and from this it is immediate that this class of morphism is pullback stable.

Example 36 *(Open/closed subobjects.) For any $a : X \to \mathbb{S}$, $a^*(i) \hookrightarrow X$ is a triquotient inclusion, $i = 0, 1 : 1 \to \mathbb{S}$. We will prove this for $i = 1$, i.e. open subobjects only. Let $\ulcorner \sqcap \urcorner : \mathbb{S} \to \mathbb{S}^{\mathbb{S}}$ be the exponential transpose of $\sqcap : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$ taken in $[\mathbf{C}^{op}, \mathbf{Set}]$. Certainly $\mathbb{S}^1 \ulcorner \sqcap \urcorner = Id$, but any $\beta : \mathbb{S} \to \mathbb{S}$ is a weak triquotient assignment on the identity and so $\beta(1) \sqcap j \sqsubseteq \beta(j)$ for any j in $\mathbb{S}$, i.e. $\ulcorner \sqcap \urcorner \mathbb{S}^1 \sqsubseteq Id$. Finally*

$$\ulcorner \sqcap \urcorner (j) \sqcap \beta \sqsubseteq_{\mathbb{S}^{\mathbb{S}}} \ulcorner \sqcap \urcorner (j \sqcap \beta(1))$$

since $k \sqcap j \sqcap \beta(k) \sqsubseteq k \sqcap j \sqcap \beta(1)$ for any k. Thus $1 : 1 \to \mathbb{S}$ is an open map and $\mathbb{S}^1 \ulcorner \sqcap \urcorner = Id$, and hence is a triquotient inclusion. But this is pullback stable and so every $a^*(1) \hookrightarrow \mathbb{S}$ is a triquotient inclusion. We will see below that every open map with $\mathbb{S}^p p_\# = Id$ is of this form. Identically, using an order dual proof, $b^*(0) \hookrightarrow \mathbb{S}$ is a triquotient inclusion for any $b : X \to \mathbb{S}$

Example 37 *Any subobject of the form $a^*(1) \wedge_{Sub(X)} b^*(0) \hookrightarrow X$ is a triquotient inclusion by pullback stability. This is the most general class available using closure of closed and open subobjects under pullback, since closed and open subobjects are, individually, closed under finite intersection.*

We now show that every triquotient inclusion is a regular monomorphism of the form $a^*(1) \wedge_{Sub(X)} b^*(0)$.

Lemma 38 *If $p : Z \to X$ is a triquotient inclusion then there exists $a, b : X \to \mathbb{S}$ such that $p : Z \to X \cong a^*(1) \wedge_{Sub(X)} b^*(0) \hookrightarrow X$.*

Proof. Set $a = p_\#(1)$ and $b = p_\#(0)$. Certainly $p : Z \to X$ then factors via $i : a^*(1) \wedge_{Sub(X)} b^*(0) \hookrightarrow X$ since $\mathbb{S}^p p_\# = Id$ (evaluated at 1 and 0) implies

$$\begin{array}{ccc} Z & \overset{!^Z}{\to} & 1 \\ p\downarrow & & \downarrow 1 \\ X & \overset{a}{\to} & Z \end{array}$$

commutes and similarly with b and 0 in the place of a and 1. Hence p factors through, say, $k : Z \to a^*(1) \wedge_{Sub(X)} b^*(0)$. Now $\mathbb{S}^p p_\# = Id$ implies

$$\mathbb{S}^k(\mathbb{S}^i p_\#) = Id$$

and so it remains to verify that

$$(\mathbb{S}^i p_\#)\mathbb{S}^k = Id.$$

But $\mathbb{S}^i$ is an epimorphism by Axiom 5 and so it is sufficient to prove $\mathbb{S}^i = \mathbb{S}^i p_\# \mathbb{S}^p$. But since $p_\#$ is a weak triquotient assignment we have that

$$p_\# \mathbb{S}^p = (p_\#(1) \sqcap (\_)) \sqcup p_\#(0)$$

and so $\mathbb{S}^i = \mathbb{S}^i p_\# \mathbb{S}^p$ since $\mathbb{S}^i p_\#(1) = 1$, $\mathbb{S}^i p_\#(0) = 0$ by construction of i and the fact that $\mathbb{S}^i$ preserves finite meets and joins. ■

Exactly as in our discussion of proper and open regular epimorphisms we have that every proper regular monomorphism is of the form $b^*(0) \hookrightarrow X$ and every open regular monomorphism is of the form $a^*(1) \hookrightarrow X$. To see this apply the lemma and note that if $p : Z \to Y$ is proper and a regular monomorphism then from $\mathbb{S}^p \forall_p \mathbb{S}^p = \mathbb{S}^p$ we have that $\mathbb{S}^p \forall_p = Id$ since, Axiom 5, $\mathbb{S}^p$ is an epimorphism.

Also mimicking our discussion of regular epimorphisms it is not true in general that every regular monomorphism in **C** is a triquotient inclusion. In **Loc** it is not the case that every sublocale is the meet of an open and closed sublocale, and so this is clear.

The following table summarizes the definitions of classes of subobjects in terms of weak triquotient assignments $p_\#$.

Class	Conditions on w.t.a. $p_\#$	Equivalent Charact.	Cond. on $\beta : S_Y^{Z_p} \to S_Y$
Triq. incl.	$\mathbb{S}^p p_\# = Id$	$a^*(1) \wedge b^*(0)$	$\mathbb{S}_Y^{!^{Z_p}} \beta = Id$
Closed sub.	$\mathbb{S}^p p_\# = Id$ $p_\#(1) = 1$	$b^*(0)$	$\mathbb{S}_Y^{!^{Z_p}} \beta = Id$ & $Id \sqsubseteq \beta \mathbb{S}_Y^{!^{Z_p}}$
Open sub.	$\mathbb{S}^p p_\# = Id$ $p_\#(0) = 0$	$a^*(1)$	$\mathbb{S}_Y^{!^{Z_p}} \beta = Id$ & $\beta \mathbb{S}_Y^{!^{Z_p}} \sqsubseteq Id$

12 Image Factorization and Compact Hausdorff objects

In this section we give an axiomatic account of the regularity of the category of compact Hausdorff objects. Classically this result is a well known consequence of Manes' theorem on the mondaicty of compact Hausdorff spaces over **Set**, see for example III 2.4 of [Johnstone 82].

Lemma 39 *Every proper map $f : X \to Y$ can be factored as $f : X \stackrel{q}{\twoheadrightarrow} a^*(0) \stackrel{i_{a^*(0)}}{\hookrightarrow} Y$ where q is a proper surjection and $a^*(0) \hookrightarrow Y$ is a closed subspace.*

Proof. Take $a = \forall_f(0)$ and so f factors via $i : a^*(0) \hookrightarrow Y$ as in the characterization of triquotient inclusions above (applicable since $\mathbb{S}^f \forall_f \sqsubseteq Id$ and so $\mathbb{S}^f \forall_f(0) = 0$). To complete the proof it needs to be verified that q is a proper surjection. Define $\forall_q$ to be the composite $\mathbb{S}^X \stackrel{\forall_f}{\to} \mathbb{S}^Y \stackrel{\mathbb{S}^i}{\twoheadrightarrow} \mathbb{S}^{a^*(0)}$. Proving that q is proper (via $\forall_q$) is then routine provided we know that $\mathbb{S}^i \times Id_{\mathbb{S}^X} = \mathbb{S}^{i+1_X}$ is an epimorphism. By Axiom 5 it is sufficient to show that $i+1 : a^*(0)+X \to Y+X$ is a regular monic. But pullback along $0_{\mathbb{S}} : 1 \to \mathbb{S}$ preserves coproduct by Axiom 2 and so $i+1$ is the pullback of $0_{\mathbb{S}} : 1 \to \mathbb{S}$ along $Y + X \stackrel{[a,0_{\mathbb{S}}]}{\to} \mathbb{S}$ proving that $i+1$ is a regular monic.

That $\mathbb{S}^q \forall_q \sqsubseteq Id$ is immediate since $\mathbb{S}^f \forall_f \sqsubseteq Id$. ■

It is not possible to use a weak triquotient assignment to obtain a similar factorization, as this would imply that every $f : X \to Y$ factors through the initial object 0 due to the existence of trivial weak triquotient assignments.

Corollary 40 *Any proper monomorphism $f : X \to Y$ is a closed subobject.*

Proof. The kernel pair of f is the same as the kernel pair of q where q is the factorization of f given in the lemma. But if f is a monomorphism its kernel pair is a pair of identity morphisms. q coequalizers its kernel pair and so must be an isomorphism. ■

An object in **C** is *strongly Hausdorff* if its diagonal is proper. So X is strongly Hausdorff exactly when $\Delta : X \to X \times X$ is open in $\mathbf{C}^{co}$; in other words, strong Hausdorffness is order dual to having an open diagonal. Vermeulen shows in [Vermeulen 91] that a locale is compact and strongly Hausdorff if and only if it is compact regular. Compact regularity is therefore order dual to discrete.

We can show, axiomatically, that,

Theorem 41 *(a) the full subcategory of compact and strongly Hausdorff objects is regular.*

(b) the full subcategory of discrete objects is regular.

Proof. (a) It is an easy categorical exercise, using the pullback stability of proper maps, to prove that compact strongly Hausdorff objects are closed under finite limits. Note that any $f : X \to Y$ factors as $f = X \overset{(1,f)}{\to} X \times Y \overset{\pi_2}{\to} Y$ and so is proper if X is compact and Y is strongly Hausdorff. The lemma and corollary prove pullback stable image factorization.

(b) Order dual proof. ∎

The category of discrete locales is equivalent to the category of discrete topological spaces; that is, the category of sets. Classically (for example, assuming the prime ideal theorem) the category of compact Hausdorff spaces is equivalent to the category of compact regular locales. Therefore, up to the categorical structure definable via regularity (e.g. image factorization, relational composition), the category of compact Hausdorff spaces *is the same theory* as the category of sets. That this categorical structure is the same was key to [Townsend 96] where compact Hausdorff relational composition is then developed in parallel to set theoretic relational composition to prove information system representation type results.

Taylor, in [Taylor 00], was the first to make this connection between compact Hausdorff and discrete precise using categorical logic; indeed he goes further showing the same result up to pretopos structure. We hope to make the relationship between Taylor's Abstract Stone Duality approach and the approach offered here the subject of further work.

13 The Double Power Monad

In this section we strengthen Axiom 4 and show how this leads to a categorical proof of Vickers' result that the category localic locales internal to the category of localic locales is equivalent to the category of locales, [Vickers 01b]. Let us concentrate only on $Z = 1$; for the same results relative to a general Z, argue relative to the topos $Sh(Z)$. The definition and motivation for the study of localic locales is given below in terms of the double power locale construction. Our first aim is therefore to show how a strengthening of Axiom 4 leads naturally to an axiomatic account of the double power locale (studied, for example, in [Vickers 02]).

Axiom 42 *The contravariant functor $\mathbb{S}^{(-)} : \mathbf{C} \to \mathbf{C}_1^{op}$ has a right adjoint.*

Note that we are not claiming that this strengthening of Axiom 4 (a) is strict. Let $G : \mathbf{C}_1^{op} \to \mathbf{C}$ denote the right adjoint then it can be verified that,

Lemma 43 *Given this extra axiom, for any object X of $\mathbf{C}$, the representable functor $\mathbf{C}(\_, G\mathbb{S}^X)$ is the exponential $\mathbf{C}(\_, \mathbb{S})^{\mathbb{S}^X}$ in $[\mathbf{C}^{op}, \mathbf{Set}]$. Conversely if such a double exponential exists for every object X, and is representable, then $\mathbb{S}^{(\_)}$: $\mathbf{C} \to \mathbf{C}_1^{op}$ has a right adjoint.*

We now verify the new axiom for $\mathbf{C} = \mathbf{Loc}$. Recall the double power locale construction, denoted $\mathbb{P}$, as introduced in [JoVic 91]. It is usually defined using a frame presentation:

$$\Omega\mathbb{P}X \equiv \mathbf{Fr}\langle \boxtimes a,\, a \in \Omega X \mid \vee^{\uparrow}\{\boxtimes i \mid i \in I\} = \boxtimes \vee^{\uparrow} I, \quad \forall I \subseteq^{\uparrow} \Omega X\rangle.$$

Equivalently, $\Omega\mathbb{P}$ is the left adjoint to the forgeful functor from the category of frames to the category of dcpos. For any locale W we have that

$$\begin{aligned} \mathbf{Loc}(W, \mathbb{P}X) &\cong \mathbf{Fr}(\Omega\mathbb{P}X, \Omega W) \\ &\cong \mathbf{dcpo}(\Omega X, \Omega W) \\ &\cong Nat[\mathbb{S}^X, \mathbb{S}^W] \\ &\cong Nat[\mathbf{Loc}(\_, W) \times \mathbb{S}^X, \mathbb{S}] \end{aligned}$$

which is sufficient to show that $\mathbb{S}^{\mathbb{S}^X}$ exists and is naturally isomorphic to $\mathbf{Loc}(\_, \mathbb{P}X)$. The axiom is therefore verified for $\mathbf{C} = \mathbf{Loc}$. That the double exponential $\mathbf{Loc}(\_, \mathbb{S})^{\mathbb{S}^X}$ exists as a representable functor is the main result of [TowVic 02].

Thus given the above axiom there exists an endofunctor $\mathbb{P} : \mathbf{C} \to \mathbf{C}$, which is then clearly the functor part of a monad as $\mathbb{P}$ can be described using a double exponential. We shall call this the *double power functor* and the induced monad, $(\mathbb{P}, \eta, \mu)$, the *double power monad.*

Vickers, in [Vickers 01b], introduced the category of localic locales as the opposite of the category of algebras of the double power locale. The paper shows that any algebra of the double power locale monad is an order internal distributive lattice in the category of locales. In other words they are localic distributive lattices and so, topologically, their investigation forms part of our understanding of topological lattice theory. Since these distributive lattices are *order* internal it follows that their study, conceptually at least, can be viewed as the study of *internal frames* (as the category of locales is dcpo enriched). The category of localic locales, the opposite of the category of frames, is therefore modelling, intuitively at least, a category of locales internal to the category of locales.

Since the double power locale functor also acts as a monad on the category of localic locales, we can introduce a category of localic locales relative to the catgory of localic locales and so on. Vickers established that localic locales relative to localic locales are equivalent to the category of locales, and so this process stops after two steps. The aim for the rest of this section is to reprove this result axiomatically. First let us note that axiomatically,

Lemma 44 *For any $(X, \mathbb{P}X \xrightarrow{\alpha} X)$ an algebra of the double power functor, X is an order internal distributive lattice in $\mathbf{C}$.*

Proof. $\mathbb{P}X$ is an order internal distributive lattice as it inherits this from $\mathbb{S}$. The property of being an order internal distributive lattice is closed under retracts and X is a retract of $\mathbb{P}X$ by definition of being a $\mathbb{P}$-algebra. ■

Definition 45 *(i)* $\mathbf{C}-\mathbf{C}$*, the category of* $\mathbf{C}$ objects relative to $\mathbf{C}$*, is defined as* $(\mathbf{C}^{\mathbb{P}})^{op}$*, the opposite of the algebras of the double power monad.*

(ii) The functor $\mathbf{C}{-}\mathbb{P} : \mathbf{C}-\mathbf{C} \to \mathbf{C}-\mathbf{C}$ *is defined as the opposite of the composite* $\mathbf{C}^{\mathbb{P}} \xrightarrow{U} \mathbf{C} \xrightarrow{\mathbb{P}} \mathbf{C}^{\mathbb{P}}$ *where* U *is the forgetful functor.*

Any monad defines a comonad on its category of algebras, and so $\mathbf{C}{-}\mathbb{P}$ defines a monad on $\mathbf{C}-\mathbf{C}$. Before we state and prove the main result result note that $\mathbb{P}:\mathbf{C}\to\mathbf{C}$ preserves coreflexive equalizers; this is because of Axiom 5 which shows that $\mathbb{S}^{(-)}$ takes coreflexive equalizers to coequalizers (i.e. equalizers in $\mathbf{C}_1$) and the fact that $\mathbb{P} = G\mathbb{S}^{(-)}$ where G is a right adjoint. Further,

Lemma 46 $\mathbb{P}:\mathbf{C}\to\mathbf{C}$ *is conservative (i.e. faithful and reflects isomorphisms).*

Proof. *Reflects isomorphisms.* If $\mathbb{S}^X \xrightarrow{\boxtimes_X} \mathbb{S}^{\mathbb{P}X}$ is defined as the double exponential transpose of the identity $\mathbb{P}X \to \mathbb{S}^{\mathbb{S}^X}$ then the identity on $\mathbb{S}^X$ factors as

$$\mathbb{S}^X \xrightarrow{\boxtimes_X} \mathbb{S}^{\mathbb{P}X} \xrightarrow{\mathbb{S}^{\eta_X}} \mathbb{S}^X.$$

If $f : X \to Y$ is such that $\mathbb{P}(f)$ is an isomorphism then $\mathbb{S}^{\mathbb{P}(f)} : \mathbb{S}^{\mathbb{P}Y} \to \mathbb{S}^{\mathbb{P}X}$ is an isomorphism in $[\mathbf{C}^{op}, \mathbf{Set}]$. But then by naturality of $\boxtimes$ and η we have that $\mathbb{S}^f : \mathbb{S}^Y \to \mathbb{S}^X$ is an isomorphism in $[\mathbf{C}^{op}, \mathbf{Set}]$. Therefore f is an isomorphism by Axiom 4.

Faithfulness. Given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ with $\mathbb{P}(f) = \mathbb{P}(g)$ we certainly have that $\mathbb{S}^{\mathbb{P}f} = \mathbb{S}^{\mathbb{P}g}$. Then use the naturality of $\boxtimes$ and the fact that $\boxtimes$ is a split monic to prove that $\mathbb{S}^f = \mathbb{S}^g$. It has been commented after the introduction of Axiom 5 that $\mathbb{S}^{(-)}$ is faithful and so we are done. ■

Theorem 47 $\mathbf{C} \simeq [(\mathbf{C}-\mathbf{C})^{\mathbf{C}-\mathbb{P}}]^{op}$.

Proof. $\mathbf{C}-\mathbf{C} \equiv (\mathbf{C}^{\mathbb{P}})^{op}$, equivalently $(\mathbf{C}-\mathbf{C})^{op} \cong \mathbf{C}^{\mathbb{P}}$. So we have an adjunction $\mathbb{P}:\mathbf{C}\to(\mathbf{C}-\mathbf{C})^{op}$ with $\mathbb{P}$ left adjoint to U.

Now taking the opposite, $\mathbb{P}^{op} : \mathbf{C}^{op} \to \mathbf{C}-\mathbf{C}$ is right adjoint to U^{op}, and $\mathbf{C}{-}\mathbb{P} = \mathbb{P}^{op}\circ U^{op}$ is the monad on $\mathbf{C}-\mathbf{C}$ that defines the category of co co-objects. To show that $\mathbf{C} \simeq [(\mathbf{C}-\mathbf{C})^{\mathbf{C}-\mathbb{P}}]^{op}$, we need that $(\mathbf{C}-\mathbf{C})^{\mathbf{C}-\mathbb{P}} \simeq \mathbf{C}^{op}$. In other words we are simply checking that $\mathbb{P}^{op} : \mathbf{C}^{op} \to \mathbf{C}-\mathbf{C}$ is monadic.

Let us use the reflexive coequalizer form of Beck's theorem, e.g. [Johnstone 02] A1.1.2, though note that the U there is our $\mathbb{P}^{op}$:

(i) $\mathbb{P}^{op}$ has a left adjoint,

(ii) $\mathbb{P}^{op}$ is conservative (i.e. faithful and reflect isomorphisms) and

(iii) $\mathbf{C}^{op}$ has and $\mathbb{P}^{op} : \mathbf{C}^{op} \to \mathbf{C}-\mathbf{C}$ preserves coequalizers of reflexive pairs.

For (i), we have commented that the left adjoint is U^{op}. (ii) has been verified above, and since $\mathbf{C}^{\mathbb{P}} \xrightarrow{U} \mathbf{C}$ creates equalizers (iii) has therefore been verified by our comment that $\mathbb{P} : \mathbf{C} \to \mathbf{C}$ preserves coreflexive equalizers. ■

14 Summary and Further Work

The emphasis has been on the abstract modelling of Scott continuous maps as natural transformations. This seems to go quite far, with a number of the deeper results in locale theory becoming available axiomatically: pullback stability of proper and open maps, image factorization of maps between compact (strongly) Hausdorff locales and the fact that triquotient maps are of effective descent. The parallel between proper and open is also observed as a formal duality.

However there are large parts of locale theory that do not appear to be readily expressible using only the interaction between Scott continuous maps and locale maps. For example, the construction of spectral locales and the Stone-Čech compactification functor.

Work is needed to relate the results back in detail to other known approaches using categorical logic, primarily [Taylor 00]. A further omission has been a discussion of Vickers' axiomatic approach, [Vickers 95]. Given the introduction of a model for the double power locale by the strenghening of Axiom 4 it is clear how to define the upper and lower power locales: they are certain sublocales of the double power locale. But it is not clear that the induced upper and lower monads are KZ/coKZ. This extra property of the monads is part of Vicker's axiomatization, and also appears to be key to the further analysis of, for example, information system theory ([Townsend 96] and [Vickers 93]).

Thus although we have finished with a good account of proper and open maps in locale theory, there is still much to do in terms of creating an entirely categorical account of locale theory. Principally there has been no attempt to give any localic constructions, but more subtly we have possibly come across a barrier blocking the parallel between proper and open. Whilst the formalization of this parallel as a duality has great appeal, we may be sacrificing some sort of topological essence should this be taken too far. After all the theory of sets is not the same thing as the theory of compact Hausdorff spaces. What is the part of topology that is not symmetric between proper and open? Can it somehow be extracted? There is therefore further work both in terms of detailed results on, for example, axiomatic information systems, and also in terms of more structural insights into the nature of topology.

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16 Appendix

This appendix gives a categorical proof of the double coverage result used to describe coequalizers in $\mathbf{DLat}(\mathbf{C})$ in terms of coequalizer in $\mathbf{C}$. The word coverage originally comes from the sheaf-theoretic idea of a coverage used to define toposes from categories of presheaves. The word was used by Johnstone (at the suggestion of S. MacLane) in his description of frame coequalizer in terms of C-ideals. The suplattice and preframe versions of the coverage theorem (in [Johnstone 82] and [JoVic 91] respectively) show how each frame coequalizer can be calculated as a particular suplattice (respectively preframe) coequalizer. A generalized result over any symmetric monoidal category, $\mathbf{C}$, was given in [Townsend 96]. This result can be applied to $\mathbf{C}$ = suplattice, preframe to give the suplattice and preframe versions (and $\mathbf{C}$ = rings to prove the ring theoretic assertion that a ring surjection can be found by Abelian group quotient). In [TowVic 02] a double coverage result is introduced which shows how the suplattice and preframe versions of the coverage theorem can be applied together in order to show how any frame coequalizer (and therefore any frame presentation by generators and relations) can be calculated as a dcpo coequalizer. We now prove a generalized double coverage theorem in the case where $\mathbf{C}$ is order enriched cartesian closed.

Theorem 48 *(double coverage) Let* $\mathbf{C}$ *be an order enriched cartesian closed category with order enriched coequalizers. If* $f, g : L \rightrightarrows N$ *is a diagram in* $\mathbf{DLat}(\mathbf{C})$ *then*

$$L \overset{f}{\underset{g}{\rightrightarrows}} N \xrightarrow{q} Q$$

is a coequalizer diagram in $\mathbf{DLat}(\mathbf{C})$ *where* q *is defined as the coequalizer of*

$$N \times N \times L \overset{1\times 1\times f}{\underset{1\times 1\times g}{\rightrightarrows}} N \times N \times N \xrightarrow{1\times\vee} N \times N \xrightarrow{\wedge} N$$

in $\mathbf{C}$.

Proof. $\mathbf{C}$ is cartesian closed and so for any object, P, the functor $(\_) \times P : \mathbf{C} \to \mathbf{C}$ preserves coequalizers since it is a left adjoint. Therefore $N \times N \xrightarrow{q\times 1} Q \times N$ coequalizes

$$(N \times N \times L) \times N \overset{r_f \times 1}{\underset{r_g \times 1}{\rightrightarrows}} N \times N$$

where $r_f = \wedge(1\times\vee)(1\times 1\times f)$ and $r_g = \wedge(1\times\vee)(1\times 1\times g)$. But $q\wedge(r_f\times 1) = q\wedge(r_g\times 1)$ and so there exists unique $a : Q\times N \to Q$ such that $a(q\times 1) = q\wedge$. But $1\times q : Q\times N \to Q\times Q$ coequalizes

$$Q\times(N\times L\times N) \underset{1\times r_g}{\overset{1\times r_f}{\rightrightarrows}} Q\times N$$

Also

$$\begin{aligned} a(1\times r_f)(q\times 1) &= a(q\times 1)(1\times r_f) \\ &= q\wedge(1\times r_f) = q\wedge(1\times r_g) \\ &= a(q\times 1)(1\times r_g) = a(1\times r_g)(q\times 1) \end{aligned}$$

and so $a(1\times r_f) = a(1\times r_g)$ since $q\times 1$ is a coequalizer. It follows that there exists unique $\wedge_Q : Q\times Q \to Q$ such that $\wedge_Q(1\times q) = a$. It must be verified that $\wedge_Q$ is right adjoint to $\Delta_Q : Q \hookrightarrow Q\times Q$. Note first that for any order enriched coequalizer $h : P_1 \twoheadrightarrow P_2$ and any object R, the bijection $(\_)\circ h : \mathbf{C}(P_2, R) \to \mathbf{C}(P_1, R)$ is necessarily an order isomorphism. So to prove $\Delta_Q\wedge_Q \sqsubseteq 1_{Q\times Q}$ it is sufficient to show that $\Delta_Q \wedge_Q (q\times 1) \sqsubseteq q\times 1$ and therefore also sufficient to show that $\Delta_Q \wedge_Q (q\times 1)(1\times q) \sqsubseteq (q\times 1)(1\times q)$. But,

$$\begin{aligned} \Delta_Q \wedge_Q (q\times 1)(1\times q) &= \Delta_Q q \wedge_N \\ &= (q\times q)\Delta_N \wedge_N \\ &\sqsubseteq (q\times q)1. \end{aligned}$$

Similarly, to establish $1_{Q\times Q} \sqsubseteq \wedge_Q\Delta_Q$, note that $q \sqsubseteq \wedge_Q\Delta_Q q$ since $\wedge_Q\Delta_Q q = q\wedge_N \Delta_N$. Define $1_Q : 1 \to Q$ by $1_Q = 1 \overset{1_N}{\to} N \overset{q}{\to} Q$. Then $!^Q 1_Q = !^Q q 1_N = !^N 1_N \sqsubseteq Id$, and $1_Q !^Q q = q 1_N !^N \sqsupseteq q Id$ and so $!^Q \dashv 1_Q$ proving that $(Q, \wedge_Q, 1_Q)$ is an order-internal meet semilattice.

Exactly similarly, by exploiting the distributivity law true of N, one can define join semilattice structure on Q. The meet and join on Q distribute since they do for N (and q is an epimorphism). Therefore $q : N \to Q$ is a map in $DLat(\mathbf{C})$. The map f factors via r_f and g factors via r_g (both through $L \overset{(1_N, Id, 0_N)}{\to} N\times L\times N$) and so $qf = qg$. Say one is given another map $N \overset{h}{\to} P$ in $DLat(\mathbf{C})$ such that $hf = hg$. Since h is a distributive lattice homomorphism it follows that $hr_f = hr_g$ and so there exists a unique map $n : Q \to P$ of $\mathbf{C}$ such that $nq = h$. To complete it must be verified that n is a distributive lattice homomorphism. But, for example, the outer square of

$$\begin{array}{ccccc} N\times N & \overset{q\times q}{\twoheadrightarrow} & Q\times Q & \overset{n\times n}{\to} & P\times P \\ \downarrow \wedge_N & & \downarrow \wedge_Q & & \downarrow \wedge_P \\ N & \overset{q}{\to} & Q & \overset{n}{\to} & P \end{array}$$

commutes as h is a meet semilattice homomorphism and so n preserves binary meet. It follows, similarly, that n is a distributive lattice homomorphism. ■

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